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Towards a non-abelian electric-magnetic symmetry: the skeleton group

L. Kampmeijer,[a] **F. A. Bais,**[a,b] **B. J. Schroers**[c] **and J. K. Slingerland**[d,e]

[a] *Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*

[b] *Santa Fe Institute, 1120 Canyon Rd, Santa Fe, NM 87501, USA*

[c] *Department of Mathematics and Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*

[d] *Dublin Institute for Advanced Studies, School for Theoretical Physics, 10 Burlington Rd, Dublin, Ireland*

[e] *Department of Mathematical Physics, National University of Ireland, Maynooth, Co. Kildare, Ireland*

E-mail: leo.kampmeijer@uva.nl, f.a.bais@uva.nl, bernd@ma.hw.ac.uk, slingerland@stp.dias.ie

Abstract: We propose a unified electric-magnetic symmetry group in Yang-Mills theory, which we call the skeleton group. We work in the context of non-abelian unbroken gauge symmetry, and provide evidence for our proposal by relating the representation theory of the skeleton group to the labelling and fusion rules of charge sectors, and by showing how the skeleton group arises naturally in a gauge-fixed description of the theory. In particular we show that the labels of electric, magnetic and dyonic sectors in non-abelian Yang-Mills theory can be interpreted in terms of irreducible representations of the skeleton group. Decomposing tensor products of these representations thus gives candidate fusion rules for these charge sectors. We demonstrate consistency of these fusion rules with the known fusion rules of the purely electric or magnetic sectors, and extract new predictions for the fusion rules of dyonic sectors in particular cases. We also implement S-duality and show that the fusion rules obtained from the skeleton group commute with S-duality. As further evidence for the relevance of the skeleton group we consider a generalisation of 't Hooft's abelian gauge fixing procedure. We show that the skeleton group plays the role of an effective symmetry in this gauge, and argue that this gauge is particularly useful for exploring phases of the theory which generalise Alice electrodynamics.

Keywords: Solitons Monopoles and Instantons, Gauge Symmetry, Confinement, Duality in Gauge Field Theories.

Contents

1. Introduction

In this paper we try to determine the electric-magnetic symmetry in a non-abelian gauge theory. This task may be formulated in many ways, varying in physical content and mathematical sophistication. Our main goal is to find a consistent large distance description of the electric, magnetic and dyonic degrees of freedom. We would like to uncover the hidden algebraic structure which governs the labelling and the fusion rules of the physical sectors in general gauge theories.

The standard literature on this subject is based on the dual symmetry proposed by Goddard, Nuyts and Olive [1]. Following earlier work of Englert and Windey on the generalised Dirac quantisation condition [2] they showed that the charges of monopoles in a theory with gauge group G take values in the weight lattice of the dual gauge group G^*, now known as the GNO or Langlands dual group. Based on this fact they came up with a bold yet attractive conjecture: monopoles transform as representations of the dual group.

Considering the fact that the Bogomolny Prasad Sommerfeld (BPS) mass formula for dyons [3, 4] is invariant under the interchange of electric and magnetic quantum numbers if the coupling constant is inverted as well, Montonen and Olive extended the GNO conjecture. Their proposal was that the strong coupling regime of some suitable quantum field theory is described by a weakly coupled theory with a similar Lagrangian but with the gauge group replaced by the GNO dual group and the coupling constant inverted [5], in other words the dual gauge symmetry is manifestly realised in the strongly coupled phase of the theory.

The non-abelian version of the Montonen-Olive conjecture has been proven by Kapustin and Witten [6] for a twisted $\mathcal{N} = 4$ super Yang-Mills theory. Using the identification of singular monopoles with 't Hooft operators and computing the operator product expansion (OPE) for the latter they showed that the fusion rules of purely magnetic monopoles are identical to the fusion rules of the dual gauge group. It was shown in [7] that the classical fusion rules of monopoles in an ordinary $\mathcal{N} = 4$ Yang-Mills theory are also consistent with the non-abelian Montonen-Olive conjecture.

A stronger version of the GNO conjecture is that a gauge theory has a hidden electric-magnetic symmetry of the type $G \times G^*$. The problem with this proposal is that the dyonic sectors do not respect this symmetry in phases where one has a residual non-abelian gauge symmetry. In such phases it may be that in a given magnetic sector there is an obstruction to the implemention of the full electric group. In a monopole background the global electric symmetry is restricted to the centraliser in G of the magnetic charge [8, 9, 10, 11, 12, 13]. Dyonic charge sectors are thus not labelled by a $G \times G^*$ representation but instead up to gauge transformations by a magnetic charge and an electric centraliser representation [14]. This interplay of electric and magnetic degrees of freedom is not captured by the $G \times G^*$ structure. Therefore one would like to find an underlying algebraic structure, reflecting the complicated pattern of the different electric-magnetic sectors in such a non-abelian phase. This algebraic structure would have to generate the complete set of fusion rules for all the different sectors, which is not known at present, and in particular would have to combine the different centraliser groups that may occur in such phases within one framework. It also has to be consistent with the fact that in the purely electric sector charges are labelled by the the full electric gauge group G, while in the purely magnetic sector, at least for the twisted $\mathcal{N} = 4$ considered by Kapustin and Witten in [6], monopoles form representations of the magnetic gauge group G^*.

As explained by Kapustin in [15] there is an equivalent labelling of dyonic charge sectors by elements in the set $(\Lambda \times \Lambda^*)/\mathcal{W}$, where $\mathcal{W}$ is the Weyl group (which isomorphic for G and G^* and Λ) and Λ^* are the weight lattices of respectively G and G^*.

Starting out from Kapustin's labelling of dyonic charge sectors and generalising an earlier proposal by two of the authors [16] we introduce the *skeleton group* S as a candidate for the electric-magnetic symmetry group in a non-abelian gauge theory. The skeleton group is in general a non-abelian group that allows one to manifestly include non-abelian electric and magnetic monopole degrees of freedom. It therefore implements (at least part of) the hidden electric-magnetic symmetry explicitly and the representation theory of S provides us with a consistent set of fusion rules for the dyonic sectors for an arbitrary gauge group. Nonetheless, it does not quite fulfill our original objective. The skeleton group has roughly the product structure $S = \mathcal{W} \ltimes (T \times T^*)$ where T and T^* are the maximal tori of G and G^*. Therefore S contains neither the full electric gauge group G nor the magnetic group G^*, and this of course implies that its representation theory will not contain the representation theories of either G or G^*. We show, however, that in the purely electric sector the representation theory of the skeleton group is consistent with the representation theory of G.

Our skeleton approach matches an interesting proposal of 't Hooft [17]. In order to get a handle on non-perturbative effects in gauge theories, like chiral symmetry breaking and confinement, 't Hooft introduced the notion of *non-propagating gauges*. An important example of such a non-propagating gauge is the so-called *abelian gauge*. In this gauge a non-abelian gauge theory can be interpreted as an abelian gauge theory (with the abelian gauge group equal to the maximal torus of G) with monopoles in it. This has led to a host of interesting approximation schemes to tackle the aforementioned non-perturbative phenomena which remain elusive from a first principle point of view, see e.g. [18, 19, 20, 21].

In this paper we propose a generalisation of 't Hooft's proposal, from an abelian to a minimally non-abelian scheme. That is where the skeleton group comes in: it plays the role of the residual symmetry in a gauge which we call the skeleton gauge. The attractive feature is that our generalisation does not affect the continuous part of the residual gauge symmetry after fixing. It is still abelian, but our generalisation adds (non-abelian) discrete components to that residual symmetry. This implies that in our skeleton gauge the non-abelian features of the gauge theory manifest themselves through topological interactions only, and that makes them manageable. The effective theories we end up with are generalisations of Alice electrodynamics [22, 23, 24]. In this sense the effective description of the non-abelian theory with gauge group G in the skeleton gauge is a merger of an abelian gauge theory and a (non-abelian) discrete gauge theory [25, 26].

The motivation for exploring non-propagating gauges is to obtain a formulation of the theory as much as possible in terms of the physically relevant degrees of freedom. In that sense 't Hooft's approach looks like studying the Higgs phase in a unitary gauge, but it goes beyond that because one does not start out from a given phase determined by a suitable (gauge invariant) order parameter. Instead, the effective theory in the abelian gauge is obtained after integrating out the non-abelian gauge field components. Nonetheless, the resulting theory is particularly suitable for describing the Coulomb phase where the residual gauge symmetry is indeed abelian. Similarly, the skeleton group is related to a generalised Alice phase.

Once this gauge-phase relation is understood our skeleton formulation not only allows us to obtain the precise fusion rules for the mixed and neutral sectors of the theory, but as a bonus allows us to analyse the phase structure of gauge theories. Yang-Mills theories give rise to confining phases, Coulomb phases, Higgs phases, discrete topological phases, Alice phases etc. These phases differ not only in their particle spectra but also in their topological structure. It is therefore crucial to have a formulation that highlights the relevant degrees of freedom, allowing one to understand what the physics of such phases is.

Starting from the skeleton gauge we are in a position to answer kinematic questions concerning different phases and possible transitions between them. For this purpose it is of the utmost importance to work in a scheme where one can compute the fusion rules involving electric, magnetic and dyonic sectors. This is deduced from some common wisdom concerning the abelian case where the fusion rules are very simple: if there is a condensate corresponding to a particle with a certain electric or magnetic charge then any particle with a multiple of this charge can consistently be thought of as absorbed by the vacuum. For confinement we know that if two electric-magnetic charges do not confine then the sum of these charges will also not confine. Given the fusion rules predicted by the skeleton group we can in principle analyse all phases that emerge from generalised Alice phases by condensation or confinement.

The outline of the paper is as follows. After introducing our conventions and notation in section 2, we explain, in section 3, the equivalence between the labelling of dyonic charge sectors involving centraliser representations and the labelling introduced by Kapustin [15]. In section 4 we introduce the skeleton group as a candidate for a unified electric-magnetic symmetry group in Yang-Mills theory. A substantial part of this section is taken up by a detailed exposition of various aspects of the skeleton group which are needed in subsequent sections. Important results are contained in section 5, where we provide evidence for the relevance of the skeleton group by relating the representation theory of the skeleton group to the labelling and fusion rules of charge sectors. In particular we show that the labels of electric, magnetic and dyonic sectors in a non-abelian Yang-Mills theory can be interpreted in terms of irreducible representations of the skeleton group. Decomposing tensor products of irreducible representations of the skeleton group thus gives candidate fusion rules for these charge sectors. We demonstrate consistency of these fusion rules with the known fusion rules of the purely electric or magnetic sectors, and extract new predictions for the fusion rules of dyonic sectors in particular cases.

One should expect the dyonic sectors and fusion rules to be robust and in particular independent on the dynamical details of the particular model. Hence, in this chapter we will not focus on special models. Nonetheless, our results must be consistent with what is known for example about S-duality of $\mathcal{N} = 4$ super Yang-Mills theories. After giving a brief review of S-duality and its action on dyonic charge sectors in section 6 we therefore show that the fusion rules obtained from the skeleton group commute with S-duality.

In section 7 we come to a final piece of evidence for the relevance of the skeleton group which goes beyond the consistency checks of the preceding sections. For this purpose we introduce the skeleton gauge which is a minimal non-abelian extension of 't Hooft's abelian gauge [17]. We argue that the skeleton group plays the role of an effective symmetry in the skeleton gauge. Moreover, we prove that the skeleton gauge incorporates intrinsically non-abelian configurations, so-called Alice fluxes, which are excluded in the abelian gauge. Hence, compared to the abelian gauge, the skeleton gauge is particularly useful for exploring non-abelian phases of the theory which generalise Alice electrodynamics [22, 23, 24] and phases, as listed at the end of the section, that emerge from generalised Alice phases by condensation or confinement. The skeleton gauge is thus necessary to reveal certain phases of the theory which are difficult to study in the abelian gauge. An important example is a novel phase we predict where particles have "lost" their charges.

2. Lie algebra conventions

We briefly summarise some facts and conventions that we shall use in the subsequent sections regarding Lie algebras and Lie groups. Additional background material can be found in e.g. [27].

By $\mathfrak{t}$ we shall denote a fixed Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of rank r. H denotes an arbitrary element in $\mathfrak{t}$. In the Cartan-Weyl basis of $\mathfrak{g}$ with respect to $\mathfrak{t}$ we have:

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad [E_\alpha, E_{-\alpha}] = \frac{2\alpha_i H_i}{\alpha_i \alpha_i} \equiv \frac{2\alpha \cdot H}{\alpha^2}, \tag{2.1}$$

where $H_1, ..., H_r$ form an orthonormal basis of $\mathfrak{t}$ with respect to the Killing form $\langle .,. \rangle$ restricted to the Cartan subalgebra. The r-dimensional vectors $\alpha = (\alpha_i)_{i=1,\dots,r}$ are nothing but the roots of $\mathfrak{g}$. We use the dot notation to denote the contraction between the indices. Also note that $\alpha^2 = \alpha \cdot \alpha$. Each root α can be interpreted as an element in $\mathfrak{t}^*$:

$$\alpha : H \in \mathfrak{t} \to \alpha(H) \in \mathbb{C}, \tag{2.2}$$

where $\alpha(H)$ defined is by

$$[H, E_\alpha] = \alpha(H) E_\alpha. \tag{2.3}$$

Instead of the basis $\{H_i\}$ for $\mathfrak{t}$ one can choose a basis of the CSA associated to the simple roots via

$$H_\alpha = 2\alpha^* \cdot H, \tag{2.4}$$

where $\alpha^* = \alpha/\alpha^2$. We now find

$$[H_\alpha, E_\beta] = 2\alpha^* \cdot \beta E_\beta \qquad [E_\alpha, E_{-\alpha}] = H_\alpha. \tag{2.5}$$

The coroots H_α span the coroot lattice $\Lambda_{cr} \subset \mathfrak{t}$ and the roots span the root lattice $\Lambda_r \subset \mathfrak{t}^*$. The dual lattice of the coroot lattice is the weight lattice $\Lambda_w \subset \mathfrak{t}^*$ of $\mathfrak{g}$ generated by the fundamental weights. The dual lattice of the root lattice is the so-called magnetic weight lattice $\Lambda_{mw} \subset \mathfrak{t}$. The weight lattice $\Lambda(G)$ of a Lie group G with Lie algebra $\mathfrak{g}$ satisfies

$$\Lambda_r \subset \Lambda(G) \subset \Lambda_w, \tag{2.6}$$

while the dual weight lattice $\Lambda^*(G)$ satisfies

$$\Lambda_{cr} \subset \Lambda^*(G) \subset \Lambda_{mw}. \tag{2.7}$$

$\Lambda^*(G)$ can be identified with the weight lattice $\Lambda(G^*)$ of GNO dual group G^* [1]. The roots of G^* correspond to the coroots of G while the fundamental weights of G^* span Λ_{mw}. These relations are summarised in table 1. This table also summarises other notational conventions that will be used in subsequent sections as well as various relations that will be discussed below.

3. Charge sectors of the theory

One of the key features of the skeleton group is that it reproduces the dyonic charge sectors of a Yang-Mills theory. To appreciate this one needs some basic understanding of the electric and magnetic charge lattices and the set of dyonic charge sectors.

3.1 Electric charge lattices

To define the electric content of a gauge theory one starts by choosing an appropriate electric charge lattice Λ. Choosing an electric charge lattice corresponds to choosing a gauge group G such that Λ equals the weight lattice $\Lambda(G)$ of G. The electric charge lattice Λ can vary from the root lattice Λ_r to the weight lattice Λ_w of $\mathfrak{g}$. This corresponds to the fact that for a fixed Lie algebra $\mathfrak{g}$ one can vary the Lie group G from $\overline{G}$ all the way to $\widetilde{G}$, where $\widetilde{G}$ is the universal covering group of G and $\overline{G}$ is the so-called adjoint group, which is the covering group divided by the center $Z(\widetilde{G})$. Note that the possible electric gauge groups are not related as subgroups but rather by taking quotients.

3.2 Magnetic charge lattices

Once the electric group G is chosen one is free to choose the magnetic spectrum as long as the generalised Dirac quantisation condition [2, 1] is respected. The magnetic spectrum is defined by fixing a magnetic charge lattice Λ^*. Just like on the electric side a choice for the magnetic charge lattice corresponds to a unique choice of a magnetic group G^* whose weight lattice $\Lambda(G^*)$ equals Λ^*. Again G^* can vary all the way from $\overline{G}^*$, the universal cover of G^*, to $\widetilde{G}^*$ which is the adjoint of G^*. This variation amounts to taking the magnetic charge lattice from the weight lattice Λ_{mw} to the root lattice Λ_{cr} of the fixed Lie algebra $\mathfrak{g}^*$ of G^*.

Even though G does neither completely fix G^* nor vice versa, the generalised quantisation condition does put restrictions on the pair (G, G^*). First of all, the roots of G^* correspond to the coroots of G. Hence, the Lie algebra $\mathfrak{g}$ of G uniquely fixes the Lie algebra $\mathfrak{g}^*$ of G^* and vice versa. The universal covering groups $\widetilde{G}$ and $\overline{G}^*$ are therefore also uniquely

magnetic	*Weyl group* $\mathcal{W}^* = \mathcal{W} \simeq$	$\widetilde{W}^*/\widetilde{D}^*$	$\simeq$	W^*/D^*	$\simeq$	$\overline{W}^*/\overline{D}^*$
		$\uparrow$		$\uparrow$		$\uparrow$
	Lift Weyl group	$\widetilde{W}^*$	$\leftarrow$	W^*	$\leftarrow$	$\overline{W}^*$
		$\cap$		$\cap$		$\cap$
	Dual gauge group	$\widetilde{G}^* = \overline{G}^*/Z^*$	$\leftarrow$	G^*	$\leftarrow$	$\overline{G}^*$
		$\cup$		$\cup$		$\cup$
	Dual torus	$\widetilde{T}^* = \mathbb{R}^r/\Lambda_w$	$\leftarrow$	$T^* = \mathbb{R}^r/\Lambda$	$\leftarrow$	$\overline{T}^* = \mathbb{R}^r/\Lambda_r$
	Dual weight lattice	$\widetilde{\Lambda}^* = \Lambda_{cr}$	$\subset$	Λ^*	$\subset$	$\overline{\Lambda}^* = \Lambda_{mw}$
electric	*Weight lattice*	$\widetilde{\Lambda} = \Lambda_w$	$\supset$	Λ	$\supset$	$\overline{\Lambda} = \Lambda_r$
	Maximal torus	$\widetilde{T} = \mathbb{R}^r/\Lambda_{cr}$	$\rightarrow$	$T = \mathbb{R}^r/\Lambda^*$	$\rightarrow$	$\overline{T} = \mathbb{R}^r/\Lambda_{mw}$
		$\cap$		$\cap$		$\cap$
	Gauge Group	$\widetilde{G}$	$\rightarrow$	G	$\rightarrow$	$\overline{G} = \widetilde{G}/Z$
		$\cup$		$\cup$		$\cup$
	Lift Weyl group	$\widetilde{W}$	$\rightarrow$	W	$\rightarrow$	$\overline{W}$
		$\downarrow$		$\downarrow$		$\downarrow$
	Weyl group $\mathcal{W} \simeq$	$\widetilde{W}/\widetilde{D}$	$\simeq$	W/D	$\simeq$	$\overline{W}/\overline{D}$

Table 1: Notational conventions and relations regarding Lie algebras, Lie groups and Weyl groups.

related. Moreover, once G is fixed, the Dirac quantisation condition tells us that the set of magnetic charges Λ^* must be a subset of $\Lambda^*(G) \subset \Lambda_{mw}$. Note that Λ_{mw} is precisely the weight lattice of the universal covering group $\overline{G}^*$ of G^*. Taking Λ^* equal to $\Lambda^*(G)$ amounts to choosing G^* to be the GNO dual group of G. We thus see that once G is fixed G^* can vary between the adjoint group $\widetilde{G}^*$ and the GNO dual group of G. Analogously, if G^* is fixed G can vary between the GNO dual of G^* and the adjoint group $\overline{G}$ without violating the generalised Dirac quantisation condition.

Unless stated otherwise we shall assume that all charges allowed by the Dirac quantisation condition occur and take G and G^* to be their respective GNO duals. Note that if the fields present in the Lagrangian are only adjoint fields and one only wants to consider smooth monopoles it is natural to restrict G and G^* to be adjoint groups.

3.3 Dyonic charge sectors

It was observed in [8, 9, 10, 11, 12, 13] that in a monopole background the global gauge symmetry is restricted to the centraliser C_g of the magnetic charge g. This implies that the charges of dyons are given by a pair (g, R_λ) where g is the usual magnetic charge corresponding to an element in the Lie algebra of G and R_λ is an irreducible representation of $C_g \subset G$. It is explained in [15] how these dyonic sectors can be relabelled in a convenient way. We shall give a brief review.

Since the magnetic charge is an element of the Lie algebra one can effectively view C_g as the residual gauge group that arises from adjoint symmetry breaking where the Lie algebra valued Higgs VEV is replaced by the magnetic charge. The Lie algebra of $\mathfrak{g}_g$ of C_g is easily determined. One can choose a gauge where the magnetic charge lies in the CSA of G. Note that this does not fix g uniquely since the intersection of its gauge orbit and the CSA corresponds to a complete Weyl orbit. Now since the generators H_α of the CSA commute one immediately finds that the complete CSA of G is contained in the Lie algebra of C_g. The remaining basis elements of $\mathfrak{g}_g$ are given by E_α with α perpendicular to g. This follows from the fact that $[E_\alpha, H_\beta] = 2(\alpha \cdot \beta)/\beta^2 E_\alpha$. We thus see that the weight lattice of C_g is identical to the weight lattice of G, whereas the roots of C_g are a subset of the roots of G. Consequently the Weyl group $\mathcal{W}_g$ of C_g is the subgroup in the Weyl group $\mathcal{W}$ of G generated by the reflections in the hyperplanes perpendicular to the roots of C_g. An irreducible representation R_λ of C_g is uniquely labelled by a Weyl orbit $[\lambda]$ in the weight lattice of C_g. Hence such a representation is fixed by a $\mathcal{W}_g$ orbit in the weight lattice of G. Remembering that g itself is fixed up to Weyl transformations, and using $C_g \simeq C_{w(g)}$ for all $w \in \mathcal{W}$ we find that (R_λ, g) is uniquely fixed by an equivalence class $[\lambda, g]$ under the diagonal action of W.

One of the goals of this paper is to find the fusion rules of dyons. We have explained that dyons are classified by an equivalence class of pairs $(\lambda, g) \in \Lambda(G) \times \Lambda(G^*)$ under the action of $\mathcal{W}$. By fusion rules we mean a set of rules of the form:

$$(R_{\lambda_1}, g_1) \otimes (R_{\lambda_2}, g_2) = \bigoplus_{[\lambda,g]} N^{\lambda,g}_{\lambda_1,\lambda_2,g_1,g_2} (R_\lambda, g), \tag{3.1}$$

where the coefficients $N^{\lambda,g}_{\lambda_1,\lambda_2,g_1,g_2}$ are positive or vanishing integers. These integers are non-vanishing only for a finite number of terms. One may also expect the product in equation (3.1) to be commutative and associative. Finally one should expect that the fusion rules of G and G^* are respected for at least the purely electric and the purely magnetic cases.

4. Skeleton Group

In an abelian gauge theory with gauge group T the global electric symmetry is not restricted by any monopole background. For a non-abelian gauge theory with gauge group G the global electric symmetry that can be realised in a monopole background always contains the maximal torus T generated by the CSA of G. The magnetic charges can be identified with representations of the dual torus T^*. Hence the electric-magnetic symmetry governing the gauge theory must contain $T \times T^*$. In the abelian case $T \times T^*$ is the complete electric-magnetic symmetry group whereas in the non-abelian case we expect there to be a larger, non-abelian group containing $T \times T^*$. In this section we will define such a group, and call it the skeleton group. It is constructed using the Weyl group action on $T \times T^*$ in such a way that its irreducible representations can be mapped to the magnetic, electric and dyonic charge sectors of non-abelian gauge theory, and that its electric subgroup is manifestly a subgroup of G.

4.1 Maximal torus and its dual

The maximal torus T is the maximal abelian subgroup of G generated by $\mathfrak{t}$. There is also a well known definition of T which is slightly different but nevertheless equivalent. This definition can immediately be extended to give a clear definition of T^*. Finally this alternative description allows us to give a straightforward definition of the Weyl group action on T and T^* as we will discuss in section 4.2.

In section 2 we considered $\mathfrak{t}$ as vector spaces over $\mathbb{C}$. However, if one declares the basis $\{H_\alpha\}$ of $\mathfrak{t}$ to be real, the real span of this basis defines a real vector space $\mathfrak{t}_\mathbb{R}$. Since any element $t \in T$ can be written as $\exp(2\pi i H)$ there is a surjective homomorphism

$$H \in \mathfrak{t}_\mathbb{R} \mapsto \exp(2\pi i H) \in T. \tag{4.1}$$

The kernel of this map is the set $\Lambda^*(G)$ and there is an isomorphism

$$T \sim \mathfrak{t}_\mathbb{R}/\Lambda^*(G). \tag{4.2}$$

As a nice consistency check of this isomorphism one can consider the irreducible representations and one will indeed find that for $\mathfrak{t}_\mathbb{R}/\Lambda^*(G)$ these are labelled by elements of $\Lambda(G)$.

The dual torus T^* is by definition a maximal abelian subgroup of G^*. Recall that the coroots of G^* can be identified with the roots of G. It follows immediately that the real span of these coroots of G^* can be identified with the real span of the roots of G. This last vector space is $\mathfrak{t}^*_\mathbb{R}$ where $\mathfrak{t}$ the CSA of G. Applying the isomorphism equivalent to the map in equation (4.1) we now find that T^* is isomorphic $\mathfrak{t}^*_\mathbb{R}/\Lambda^*(G^*)$. For the special case that G^* equals the GNO dual of G so that $\Lambda^*(G^*) = \Lambda(G)$ we find that

$$T^* \sim \mathfrak{t}^*_\mathbb{R}/\Lambda(G), \tag{4.3}$$

which is consistent with the fact that the irreducible representations of the GNO dual group are labelled by elements of $\Lambda^*(G)$.

A convenient way to represent T is as follows. Let $\widetilde{G}$ be the universal cover of G. The dual weight lattice $\Lambda^*(\widetilde{G})$ for $\widetilde{G}$ equals the coroot lattice Λ_{cr}. A basis of this lattice is the set of coroots $\{H_{\alpha_i}\}$ where α_i are the simple roots of G. One thus finds that $T_{\widetilde{G}}$ is explicitly parametrised by the set $\{H = \sum_{i=1}^r \theta_i H_{\alpha_i} \in \mathfrak{t}_\mathbb{R} \mid \theta_i \in [0, 2\pi)\}$. Using the homomorphism from $\mathfrak{t}_{\mathbb{R}}$ to T from equation (4.1) we thus find that each element in $T_{\widetilde{G}}$ can uniquely be written as

$$\exp\left(i\theta_i H_{\alpha_i}\right) \tag{4.4}$$

with $\theta_i \in [0, 2\pi)$. If G does not equal its universal covering group, equation (4.4) does not provide a unique parametrisation of T in the sense that one still has to mod out the discrete group

$$Z = \Lambda^*(G)/\Lambda_{cr} \subset T_{\widetilde{G}}. \tag{4.5}$$

This follows from the fact that $G = \widetilde{G}/Z$ and hence $T_G = T_{\widetilde{G}}/Z$.

Using analogous arguments we find that any element in T^* can uniquely be represented as $H^* = \sum_{i=1}^{r} \theta_i^* H_{\alpha_i^*}$ up to an element in a discrete group Z^*. If G^* equals the GNO dual of G, Z^* is given by $\Lambda(G)/\Lambda_r$.

4.2 Weyl group action

The semi-direct product that plays a role in the definition of the skeleton group is defined with respect to the action of the Weyl group on the maximal torus of G and its dual torus. We shall briefly discuss this action.

The Weyl group is a subgroup of the automorphism group of the root system generated by the Weyl reflections

$$w_\alpha : \beta \mapsto \beta - \frac{2\alpha \cdot \beta}{\alpha^2}\alpha. \tag{4.6}$$

By linearity the action of the Weyl group can be extended to the whole root lattice, the weight lattice and $\mathfrak{t}^*$.

$$w_\alpha : \lambda \mapsto \lambda - \frac{2\alpha \cdot \lambda}{\alpha^2}\alpha. \tag{4.7}$$

Note that w_α simply corresponds to the reflection in the hyperplane in $\mathfrak{t}^*$ orthonormal to the root α.

Remember that $\mathfrak{t}^*$ is the dual space of $\mathfrak{t}$, the CSA of G. The action of $w \in \mathcal{W}$ on $H \in \mathfrak{t}$ is defined by $\alpha(w(H)) = w^{-1}(\alpha)(H)$. From this relation one finds

$$w_\alpha(H) = H - \frac{2\langle H, H_\alpha\rangle}{\langle H_\alpha, H_\alpha\rangle}H_\alpha, \tag{4.8}$$

where $\langle .,. \rangle$ is the Killing form, restricted to the Cartan subalgebra. In particular one finds

$$w_\alpha(\beta^*) = \beta^* - \frac{2\beta^* \cdot \alpha^*}{(\alpha^*)^2}\alpha^* \tag{4.9}$$

and

$$w^{-1}(H_\alpha) = w^{-1}(2\alpha^* \cdot H) = 2w(\alpha^*) \cdot H = H_{w(\alpha)}. \tag{4.10}$$

The action of the Weyl group on $\mathfrak{t}$ induces an action on T as follows

$$w \in \mathcal{W} : \exp\left(i\theta_i H_{\alpha_i}\right) \in T \mapsto \exp\left(i\theta_i H_{w(\alpha_i)}\right) \in T. \tag{4.11}$$

Analogously one can define the action of the Weyl group on the dual torus:

$$w \in \mathcal{W} : \exp\left(i\theta_i^* H_{\alpha_i^*}\right) \in T^* \mapsto \exp\left(i\theta_i^* H_{w(\alpha_i^*)}\right) \in T^*. \tag{4.12}$$

4.3 Definition of the skeleton group

The definition of the skeleton group in this section should incorporate the actions of the Weyl group on T and T^* defined in the previous section. Hence, one might define the skeleton group as the semi-direct product $\mathcal{W} \ltimes (T \times T^*)$, with the diagonal action of $\mathcal{W}$ on $T \times T^*$. It is difficult to give a direct physical interpretation to this group, since its electric subgroup $\mathcal{W} \ltimes T$ and in particular the Weyl group $\mathcal{W} \subset \mathcal{W} \ltimes T$, is usually not a subgroup of the gauge group G. However, recall that the Weyl group is isomorphic to the normaliser of T in G modulo the centraliser of T. In fact, the Weyl group can be lifted to G, as we shall explain below. Similarly, since the Weyl group only depends on the Lie algebra, it should not be very surprising that the Weyl group can actually be lifted to any Lie group with this fixed algebra. We will use this fact to show that we can define the skeleton group such that its electric part is indeed a subgroup of G.

According to [28, 29], a natural finite lift $\overline{W}$ of $\mathcal{W}$ into the group of automorphisms of $\mathfrak{g}$ is defined as follows. For any simple root α of G, we define a lift $\overline{w}_\alpha$ of the Weyl reflection w_α by

$$\overline{w}_\alpha = \mathrm{Ad}(x_\alpha) \tag{4.13}$$

with

$$x_\alpha = \exp\left(\frac{i\pi}{2}(E_\alpha + E_{-\alpha})\right). \tag{4.14}$$

The $\overline{w}_\alpha$ generate $\overline{W}$, which is a finite subgroup of the automorphism group of $\mathfrak{g}$. Note that $\overline{W}$ is also a subgroup of the adjoint group $\overline{G}$ of G. $\overline{W}$ has an abelian normal subgroup $\overline{D}$ generated by the elements $\overline{w}_\alpha^2$ and we have $\mathcal{W} = \overline{W}/\overline{D}$.

If one wants to lift $\mathcal{W}$ into the group G itself, rather than into its adjoint representation, one can do this by lifting $\overline{W} \subset G/Z_G$ into G. Such a lift W' of $\mathcal{W}$ can be defined as the preimage of $\overline{W}$ under the projection from G to its adjoint group G/Z_G. Alternatively, one can define a lift W of $\mathcal{W}$ into G as the group generated by the elements x_α of G. In general, we might have $W \neq W'$, although it is clear that $W \subset W'$. In the remainder of this paper we shall ignore this possible subtlety and only consider the lift W. We shall also use the abelian normal subgroup $D \subset W$ defined by $D = W \cap T$.

We now introduce the skeleton group S as

$$S = (W \ltimes (T \times T^*))\,/D, \tag{4.15}$$

where the action of $d \in D$ is by simultaneous left multiplication on $W \ltimes T$. The action of W on the two maximal tori is the usual conjugation action and it factors over the quotient $\mathcal{W}$ of W, i.e. every element $w \in W$ acts just like the corresponding element of the Weyl group $\mathcal{W}$. Note that equivalently we can write:

$$S = \frac{W \ltimes T}{D} \ltimes T^*. \tag{4.16}$$

We define the electric subgroup S_{el} of S as

$$S_{el} = \{s \in S \mid s = (w, t, 1)D,\ w \in W,\ t \in T\}. \tag{4.17}$$

One may now define $\phi : W \ltimes T \to G$ by

$$\phi(w,t) = wt^{-1}. \tag{4.18}$$

It is easy to check that ϕ is a homomorphism into $N_T \subset G$, the normaliser of T. The kernel of ϕ is precisely the set of elements $(d,d) \in W \ltimes T$, with necessarily $d \in D$. As a result, S_{el} is isomorphic to the image of ϕ, which is in turn a subgroup of $N_T \subset G$ and we have achieved our goal to make the electric part of the skeleton group a subgroup of the electric group.

With the definition above one should not expect the magnetic subgroup S_{mag}, defined as

$$S_{mag} = \{s \in S \mid s = (w,1,t^*)D,\ w \in W,\ t^* \in T^*\}, \tag{4.19}$$

to be a subgroup of G^* since $S_{mag} = \mathcal{W} \ltimes T^*$ and the Weyl group $\mathcal{W}$ of G and G^* is in general not a subgroup of G^*. However, one can introduce the dual group S^* and define it to be the skeleton group of G^*. The electric subgroup S^*_{el} is then of course a subgroup of G^*.

5. Representation theory

In this section we discuss some general properties of the representations of the skeleton group and its fusion rules. We focus in particular on $SU(2)$ as an example. The general $SU(n)$ case is discussed in appendix A.

5.1 Representations of the skeleton group

The electric factor S_{el} of the skeleton group is a subgroup of G. This implies that representations of G decompose into irreducible representations of the skeleton group with trivial magnetic charges. Conversely, in the representation theory of the skeleton group only parts of G which commute with the magnetic charge are implemented. The skeleton group is thus an extension of $T \times T^*$ whose representation theory respects key features of the dyonic charge sectors. In this section we describe these properties of the skeleton group in general terms and clarify the relation with G representations. The $SU(2)$ case is worked out explicitly in section 5.3.

The representations of S correspond precisely to the representations of $W \ltimes (T \times T^*)$ whose kernel contain the normal subgroup D. Since $W \ltimes (T \times T^*)$ is a semi-direct product its irreducible representations are labelled by an orbit and a centraliser representation [30]. To be precise these orbits are subspaces in the character group of $T \times T^*$, the set of irreducible representations of $T \times T^*$, which is precisely given by $\Lambda(G) \times \Lambda(G^*)$. The diagonal action of the Weyl group on $T \times T^*$ defining the semi-direct product of the skeleton group induces a diagonal action in the character group:

$$w \in \mathcal{W} : (\lambda, g) \in \Lambda(G) \times \Lambda^*(G) \mapsto (w(\lambda), w(g)) \in \Lambda(G) \times \Lambda^*(G), \tag{5.1}$$

where

$$\begin{aligned} (w(\lambda), w(g)) : (t, t^*) \in T \times T^* \quad &\mapsto \quad \lambda(w^{-1}(t))g(w^{-1}(t^*)) \in \mathbb{C} \\ \big(\exp(i\theta_i H_{\alpha_i}), \exp(i\theta_i^* H_{\alpha_i^*})\big) \quad &\mapsto \quad \exp(i\theta_i 2w(\lambda)\cdot\alpha_i^* + i\theta_i^* 2w(g)\cdot\alpha_i). \end{aligned} \tag{5.2}$$

Here we used equations (4.11) and (4.12) together with

$$H_{w(\alpha)}|\,\lambda\,\rangle = 2\lambda\cdot w(\alpha^*)|\,\lambda\,\rangle = 2w^{-1}(\lambda)\cdot\alpha^*|\,\lambda\,\rangle, \tag{5.3}$$

and similarly

$$H_{w(\alpha^*)}|\,g\,\rangle = 2g\cdot w(\alpha)|\,g\,\rangle = 2w^{-1}(g)\cdot\alpha|\,g\,\rangle. \tag{5.4}$$

Since the lift W acts in the same way on T and T^* as $\mathcal{W}$ we now see that an irreducible representation of the skeleton group carries a label that corresponds to an $\mathcal{W}$ orbit $[\lambda, g]$ in $\Lambda(G)\times\Lambda(G^*)$. These labels are precisely the dyonic charge sectors of Kapustin [15] as discussed in section 3.3.

In order to give an explicit definition of the irreducible representations of the skeleton group let $[\lambda, g]$ denote the $\mathcal{W}$ orbit containing (λ, g) and let γ denote an irreducible representation of the centraliser $N_{(\lambda,g)}\subset W$ of (λ, g). Now for any $(\mu, h)\in[\lambda, g]$, choose some $x_{(\mu,h)}\in W$ such that $x_{(\mu,h)}(\lambda, g) = (\mu, h)$ and define $V_\gamma^{[\lambda,g]}$ to be the vector space spanned by $\{|\mu, h, e_i^\gamma\rangle\}$, where $\{e_i^\gamma\}$ is a basis for the vector space V_γ on which γ acts. Using the well known definition of an induced representation of a semi-direct product we find that the irreducible representation $\Pi_\gamma^{[\lambda,g]}$ of $W\ltimes(T\times T^*)$ acts on $V_\gamma^{[\lambda,g]}$ as follows:

$$\Pi_\gamma^{[\lambda,g]}(w, t, t^*)|\,\mu, h,\ v\,\rangle = w(\mu)(t)w(h)(t^*)|\,w(\mu), \gamma(x_{w(\mu)}^{-1}wx_\mu)v\,\rangle. \tag{5.5}$$

These representations have the attractive property that the irreducible representations of $W\ltimes(T\times T^*)$ with trivial centraliser labels are in one-to-one relation with the electric-magnetic charge sectors. However, in general not all of these representations are representations of S. The allowed representations satisfy

$$\Pi_\gamma^{[\lambda,g]}(d, d, 1)|\,\mu, h,\ v\,\rangle = |\,\mu, h,\ v\,\rangle, \tag{5.6}$$

which implies

$$d(\mu)(d)d(h)(1)|\,d(\mu), d(h), \gamma(x_{d(\mu)}^{-1}dx_\mu)v\,\rangle = |\,\mu, h,\ v\,\rangle. \tag{5.7}$$

Since $d\in T$ we have $d(\mu)=\mu$ and we find that $\Pi_\gamma^{[\lambda,g]}$ is a representation of S if

$$\mu(d)|\,\mu, h, \gamma(x_\mu^{-1}dx_\mu)v\,\rangle = |\,\mu, h,\ v\,\rangle \qquad \forall\ |\,\mu, h,\ v\,\rangle\in V_\gamma^{[\lambda,g]}. \tag{5.8}$$

This condition is satisfied if D acts trivially on all vectors of the form $|\,\lambda, v\,\rangle$. To show this we note that $\mu(t) = x_\mu(\lambda)(t) = \lambda(x_\mu^{-1}\triangleright t) = \lambda(x_\mu^{-1}tx_\mu)$. As mentioned in section 4.3 D is a normal subgroup of W, i.e. $x_\mu^{-1}dx_\mu = d'\in D$. Hence for the action of D on $|\,\mu, v\,\rangle$ we thus have

$$\mu(d)|\,\mu, \gamma(x_\mu^{-1}dx_m u)v\,\rangle = \lambda(d')|\,\mu, \gamma(d')v\,\rangle = |\,\mu\,\rangle\lambda(d')\gamma(d')|\,v\,\rangle. \tag{5.9}$$

Now if

$$\lambda(d)|\,\lambda, \gamma(d)v\,\rangle = |\,\lambda\,\rangle\lambda(d)\gamma(d)|\,v\,\rangle = |\,\lambda\,\rangle|\,v\,\rangle \tag{5.10}$$

for all $d\in D$ we find that D acts trivially on $V_\gamma^{[\lambda,g]}$.

The question that remains is if there always exists centraliser representation γ of $W_{(\lambda,g)} \subset W$ that satisfies this constraint. Note that equation (5.10) is precisely the constraint one would obtain for representations of the electric part $(W \ltimes T)/D$ of the skeleton group except that γ would be an irreducible representation of a possible larger subgroup $W_\lambda \subset W$, i.e. $W_{(\lambda,g)} \subset W_\lambda$. This means however that the restriction $\gamma_{|W_{(\lambda,g)}}$ of an allowed electric centraliser representation γ of W_λ automatically satisfies (5.10). Consequently there exists an irreducible representation of S for a given orbit $[\lambda, g]$ if there exists an irreducible representation of S_{el} for a given orbit $[\lambda]$.

It is easily seen that an irreducible representation of S_{el} labelled by $[\lambda]$ exists if λ lies in the weight lattice of G. As proven in section 4.3, S_{el} is a subgroup of G and thus all representations of the gauge group fall apart into representations of S_{el}. Moreover, both the gauge group and the skeleton group contain the maximal torus T. Hence all representations of T that appear in the restriction of G representations must also appear in the restriction of a representation of S_{el} to T. From the representation theory of G we know that all irreducible representations of T come up in this way and hence all Weyl orbits in the weight lattice of G give rise to a representation of the skeleton group. We finally note that an irreducible representation of G with highest weight λ leads to a representation of S_{el} which has a one-dimensional centraliser representation. If a representation of G has a weight with multiplicity greater than one it may give rise to an allowed centraliser representation acting on a space which has more dimensions.

5.2 Fusion rules

Here we discuss some general properties of the fusion rules of the skeleton group. We shall restrict our discussion to the electric-magnetic charges and ignore the centraliser representation for the most part. The fusion rules of the dyonic charges are found by combining the Weyl orbits of the representations. An elegant way do deal with this combinatorics is to use a group ring.

Below we define a homomorphism, denoted by "Char" from the representation ring of the skeleton group to the Weyl invariant part $\mathbb{Z}[\Lambda \times \Lambda^*]^{\mathcal{W}}$ of the group ring $\mathbb{Z}[\Lambda \times \Lambda^*]$ where $\Lambda \times \Lambda^*$ is the weight lattice of $T \times T^*$. This group ring has an additive basis given by the elements $e_{(\lambda,g)}$ with $(\lambda, g) \in \Lambda \times \Lambda^*$. The multiplication of the group ring is defined by $e_{(\lambda_1,g_1)}e_{(\lambda_2,g_2)} = e_{(\lambda_1+\lambda_2,g_1+g_2)}$. Finally the action of the Weyl group on the weight lattice induces an action on the group ring given by

$$w \in \mathcal{W} : e_{(\lambda,g)} \mapsto e_{w(\lambda),w(g)}. \tag{5.11}$$

A natural basis for the ring $\mathbb{Z}[\Lambda \times \Lambda^*]^{\mathcal{W}}$ is the set of elements of the form

$$e_{[\lambda,g]} := \sum_{(\mu,h)\in[\lambda,g]} e_{(\mu,h)}, \tag{5.12}$$

where $[\lambda, g]$ is a Weyl orbit in the weight lattice.

The homomorphism Char from the representation ring of the skeleton group to $\mathbb{Z}[\Lambda \times \Lambda^*]^{\mathcal{W}}$ is defined through mapping $|\, \mu, h, v\,\rangle \in V_\gamma^{[\lambda,g]}$ to $e_{(\mu,h)} \in \mathbb{Z}[\Lambda \times \Lambda^*]$. Consequently for an irreducible representation $\Pi_\gamma^{[\lambda,g]}$ of the skeleton group we have $e_{[\lambda,g]}$ in $\mathbb{Z}[\Lambda \times \Lambda^*]^{\mathcal{W}}$

$$\mathrm{Char} : \Pi_\gamma^{[\lambda,g]} \mapsto \dim(V_\gamma) e_{[\lambda,g]}. \tag{5.13}$$

Note that if γ is a trivial centraliser representation or some other 1-dimensional representation then Char maps to a basis element of the group algebra.

Char respects the addition and multiplication in the representation ring since

$$\mathrm{Char} : \Pi_{\gamma_1}^{[\lambda_1,g_1]} \oplus \Pi_{\gamma_2}^{[\lambda_2,g_2]} \mapsto \dim(V_{\gamma_1}) e_{[\lambda_1,g_1]} + \dim(V_{\gamma_2}) e_{[\lambda_2,g_2]} \tag{5.14}$$

$$\mathrm{Char} : \Pi_{\gamma_1}^{[\lambda_1,g_1]} \otimes \Pi_{\gamma_2}^{[\lambda_2,g_2]} \mapsto \dim(V_{\gamma_1}) \dim(V_{\gamma_2}) e_{[\lambda_1,g_1]} e_{[\lambda_2,g_2]}. \tag{5.15}$$

We can use this to retrieve the fusion rules for the dyonic charge sectors since the expansion of skeleton group representations in irreducible representations corresponds to expanding products in the Weyl invariant group ring into basis elements:

$$e_{[\lambda_1,g_1]} e_{[\lambda_2,g_2]} = \sum_{[\lambda,g]} N_{\lambda_1,\lambda_2,g_1,g_2}^{\lambda,g} e_{[\lambda,g]}. \tag{5.16}$$

If one restricts to the purely electric sector, i.e. $g = 0$, such that the centraliser $C_g \subset G$ equals G itself, one should expect to retrieve the fusion rules of G. As was noticed by Kapustin in [31] equation (5.16) does not correspond to the decomposition of tensor products of G representations. However, the fusion rules of the skeleton group also involve the centraliser representations. In particular the dimensions of the centraliser representations satisfy

$$\Pi_{\gamma_1}^{[\lambda_1,g_1]} \otimes \Pi_{\gamma_2}^{[\lambda_2,g_2]} = \bigoplus_{[\lambda,g],\gamma} \widetilde{N}_{\lambda_1,\lambda_2,g_1,g_2,\gamma_1,\gamma_2}^{\lambda,g,\gamma} \Pi_\gamma^{[\lambda,g]} \tag{5.17}$$

such that

$$\sum_\gamma \widetilde{N}_{\lambda_1,\lambda_2,g_1,g_2,\gamma_1,\gamma_2}^{\lambda,g,\gamma} \dim(V_\gamma) = \dim(V_{\gamma_1}) \dim(V_{\gamma_2}) N_{\lambda_1,\lambda_2,g_1,g_2}^{\lambda,g}. \tag{5.18}$$

If we restrict to the purely electric sector where $g = 0$ we still do not have an immediate agreement with the fusion rules for G. However, as far as it concerns the skeleton group the restriction to trivial magnetic charge gives rise to representations of S_{el}, which is a subgroup of G. This relation will be reflected in the fusion rules as we shall see for $G = SU(2)$ in section 5.3.

5.3 Fusion rules for the skeleton group of SU(2)

We shall compute the complete set of irreducible representations and their fusion rules for the skeleton group of $SU(2)$. From these fusion rules we shall find that the skeleton group is the maximal electric-magnetic symmetry group that can realised simultaneously in all dyonic charge sectors of the theory. Finally we compare our computations with similar results obtained by Kapustin and Saulina [32].

The skeleton group can be expressed as $W \ltimes (T \times T^*)$ modded out by a normal subgroup $D \subset W \ltimes T$ as explained in section 4.3. For the $SU(n)$-case W and D are computed in appendix A and for $SU(2)$ they equal respectively $\mathbb{Z}_4$ and $\mathbb{Z}_2$. The latter group is precisely the center of $SU(2)$.

The irreducible representations of S for $SU(2)$ correspond to a subset of irreducible representations of $\mathbb{Z}_4 \ltimes (T \times T^*)$ which represent D trivially. This leads to a constraint on the centraliser charges and the electric charge as given by equation (5.10).

If both the electric charge and magnetic charge vanish the centraliser is the $\mathbb{Z}_4$ which is generated by

$$x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{5.19}$$

The allowed centraliser representations are the two irreducible representations that send x^2 to 1. One of these representations is the trivial representation. This leads to the trivial representation of the skeleton group which we denote by $(+, [0,0])$. The remaining non-trivial centraliser representations maps x to -1 and gives a 1-dimensional irreducible representation of the skeleton group which we shall denote by $(-, [0,0])$.

If either the electric or the magnetic charge does not vanish the orbit under the $\mathbb{Z}_4$ action has two elements and the centraliser group is $\mathbb{Z}_2 \subset \mathbb{Z}_4$ generated by x^2. The irreducible representation of $\mathbb{Z}_2$ that satisfies equation (5.10) is uniquely fixed by the electric charge λ labelling the equivalence class $[\lambda, g]$. It is the trivial representation if the electric charge is even and it is the non-trivial representation if the electric charge is odd. We can thus denote the resulting irreducible skeleton group representation by $[\lambda, g]$ with λ or g non-vanishing. Note that these representations are 2-dimensional.

The electric-magnetic charge sectors appearing in the decomposition of a tensor product of irreducible representations of the skeleton group can be found from the fusion rules of $\mathbb{Z}[\Lambda \times \Lambda^*]$ as discussed in section 5.2. Ignoring the centraliser charges this gives the following fusion rules:

$$[0,0] \otimes [0,0] = [0,0] \tag{5.20}$$

$$[0,0] \otimes [\lambda, g] = [\lambda, g] \tag{5.21}$$

$$[\lambda_1, g_1] \otimes [\lambda_2, g_2] = [\lambda_1 + \lambda_2, g_1 + g_2] \oplus [\lambda_1 - \lambda_2, g_1 - g_2]. \tag{5.22}$$

To retrieve the fusion rules of the skeleton group itself one should take into account the centraliser representations. However, for all charges except $[0,0]$ the centraliser representations is uniquely determined. If we restrict to $[0,0]$ charges we obviously obtain $\mathbb{Z}_4$ fusion rules. This leads to:

$$(s_1, [0,0]) \otimes (s_2, [0,0]) = (s_1 s_2, [0,0]) \tag{5.23}$$

$$(s, [0,0]) \otimes [\lambda, g] = [\lambda, g] \tag{5.24}$$

$$[\lambda_1, g_1] \otimes [\lambda_2, g_2] = [\lambda_1 + \lambda_2, g_1 + g_2] \oplus [\lambda_1 - \lambda_2, g_1 - g_2]. \tag{5.25}$$

If in the last line the electric-magnetic charges are parallel so that $[0,0]$ appears at the right hand side we have to interpret this as a 2-dimensional reducible representation. Its decomposition into irreducible representations can be computed via the characters. Let $H \ltimes N$ be a general semi-direct product group with H a finite group and N abelian as is the case for $W \ltimes (T \times T^*)$. The characters of the irreducible representations $a = \Pi_\alpha^{[\sigma]}$, $b = \Pi_\beta^{[\eta]}$ and $c = \Pi_\gamma^{[\rho]}$ of $H \ltimes N$ satisfy

$$\begin{aligned}
\langle \chi_c, \chi_{a\otimes b} \rangle &= \int_{H\times N} \chi_c(h,n))\chi_a^*(h,n)\chi_b^*(h,n)dhdn \\
&= \sum_{\mu\in[\rho]}\sum_{\nu\in[\sigma]}\sum_{\zeta\in[\eta]} \delta_{\mu,\nu\zeta} \int_{H\times N} \delta_{h(\mu),\mu}\delta_{h(\nu),\nu}\delta_{h(\zeta),\zeta} \times \\
&\quad \chi_\gamma(h_\mu^{-1}hh_\mu)\chi_\alpha^*(h_\nu^{-1}hh_\nu)\chi_\beta^*(h_\zeta^{-1}hh_\zeta)dhdn.
\end{aligned} \tag{5.26}$$

Applying this formula to the skeleton group of $SU(2)$ one finds

$$[\lambda, g] \otimes [\lambda, g] = [2\lambda, 2g] \oplus (-, [0,0]) \oplus (+, [0,0]). \tag{5.27}$$

An important question is if the fusion rules obtained here provide a hint about an extended electric-magnetic symmetry. The representations of such a symmetry should be uniquely labelled by the dyonic charges and should not carry additional quantum numbers. Moreover, the representations with vanishing magnetic charge correspond to representations of the electric group. From this perspective the skeleton group representations $(\pm, [0,0])$ are part of odd dimensional representations of $SU(2)$. In this way one can at least reconstruct the fusion rules of $SU(2)$ in the magnetically neutral sector. As an example we consider equation (5.27) with λ equal the the fundamental weight of $SU(2)$ and $g = 0$:

$$[\lambda, 0] \otimes [\lambda, 0] = [2\lambda, 0] \oplus (-, [0,0]) \oplus (+, [0,0]). \tag{5.28}$$

First we identify all representations with representations of S_{el} by "forgetting" the magnetic charge. Second we note that since $S_{el} \subset SU(2)$ as proven in section 4.3 the representations of the latter fall apart into irreducible representations of the former. In particular the trivial representation of $SU(2)$ can be identified with the trivial representation of S_{el} while the triplet falls apart into $[2\lambda] \oplus (-, [0])$, with 2λ equal to the highest weight of the triplet representation. Equation (5.28) is thus a simple consequence of the fact that

$$2 \otimes 2 = 3 \oplus 1. \tag{5.29}$$

One could try to push this line of thought through for $g \neq 0$. Unfortunately, in this case $[2\lambda, 0]$ does not appear in equation (5.27) as one should expect if $SU(2)$ is contained in some extended electric-magnetic symmetry and if $(-, [0,0])$ does indeed correspond to an $SU(2)$ triplet. Adding the $[2\lambda, 0]$ term by hand readily leads to problems since this forces one to add corresponding terms on left hand side. In this case one should replace $[\lambda, g]$ by $[\lambda, g] \oplus [\lambda, -g]$. This implies that $[\lambda, g]$ could never be an irreducible representation for an extended electric-magnetic symmetry containing the skeleton group since it would have to be paired with $[\lambda, -g]$ which labels an inequivalent charge sector as discussed in section 3.3. Since extending the skeleton group seems to fail one should expect the skeleton group to be the maximal electric-magnetic symmetry group that can be realised in all dyonic charge sectors. In a set of dyonic charge sectors that is closed under fusion, such as for example the magnetically neutral sectors, the electric part of skeleton group can be extended to the centraliser group in G of the magnetic charges of that particular set of sectors. Note that since G is not fully realised in all sectors one might wonder if the skeleton group respects gauge invariance. We shall come back to that discussion in section 7.

Another approach to give a unified description of an electric group G and a magnetic group G^* is to consider the OPE algebra of mixed Wilson-'t Hooft operators. Such operators are labelled by the dyonic charge sectors as explained by Kapustin in [15]. Moreover, the OPEs of Wilson operators are given by the fusion rules of G while the OPEs for 't Hooft operators correspond to the fusion rules of G^*. These facts were used by Kapustin and Witten [6] to prove that magnetic monopoles transform as G^* representations. It is thus natural to ask what controls the product of mixed Wilson-'t Hooft operators. The answer must somehow unify the representation theory of G and G^*. Consequently one might also expect it sheds some light on the fusion rules of dyons.

For a twisted $\mathcal{N} = 4$ SYM with gauge group $SO(3)$ products of Wilson-'t Hooft operators have been computed by Kapustin and Saulina [32]. In terms of dyonic charge sectors they found for example:

$$[n,0]\cdot[0,1] = \sum_{j=0}^{n}[n-2j,1]. \tag{5.30}$$

This rule can easily be understood from the fusion rules of the skeleton group for $SO(3)$ or $SU(2)$. First we note that for $G = SO(3)$ Λ can be identified with the even integers. The magnetic weight lattice Λ^* for $G^* = SU(2)$ is then given by $\mathbb{Z}$. The $[n,0]$ sector is a magnetically neutral sector and thus corresponds to the $(n+1)$-dimensional irreducible representation of $SO(3)$ or $SU(2)$. This representation falls apart into a sum irreducible representations of S_{el}. In terms of magnetically neutral representations of the skeleton group this sum of irreducible representations is given by

$$\bigoplus_{j=0}^{n-1}[n-2j,0] + (s,[0,0]). \tag{5.31}$$

Note that the centraliser label s in (5.31) depends uniquely on n. The 't Hooft operator labelled by $[0,1]$ can be uniquely related to the irreducible representation $[0,1]$ of the skeleton group. Similarly for the Wilson-'t Hooft operators appearing at the right hand side of equation (5.30) there is also a unique identification with skeleton group representations. Finally we note that the decomposition of the tensor products of $[0,1]$ with the reducible representations (5.31) into irreducible representation of the skeleton group is given by the right hand side of equation (5.30).

A second product rule obtained in [32] which is consistent with the results of [6], can be written in terms of electrically neutral charge sectors as:

$$[0,1]\cdot[0,1] = [0,2] + [0,0]. \tag{5.32}$$

This product rule is more difficult to understand from the fusion rules of the skeleton group S of $SO(3)$. In terms of irreducible representations of S we have

$$[0,1]\otimes[0,1] = [0,2]\oplus(-,[0,0])\oplus(+,[0,0]). \tag{5.33}$$

As in the case of equation (5.28), we would like to conclude that the representations $(-,[0,0])$ and $[0,2]$ should both be part of the magnetic sector $[0,2]$, but here we cannot argue in the same way, because the magnetic part of S is not a subgroup of the magnetic group $G^* = SU(2)$. However, we can instead pass to the dual skeleton group S^* introduced in section 4.3, which is the skeleton group for $SU(2)$. The product rule (5.32) can then be identified with the $SU(2)$ tensor product decomposition given in (5.29). It is explained above that this fusion rule is consistent with the fusion rules of the skeleton group of $SU(2)$.

The last OPE product rule found in [32] can be represented as

$$[2n,1]\cdot[0,1] = [2n,2] + [2n,0] - [0,0] - [2n-2,0], \tag{5.34}$$

while from equation (5.25) we find for the related tensor product decomposition of skeleton group representations;

$$[2n,1]\cdot[0,1] = [2n,2] \oplus [2n,0]. \tag{5.35}$$

One observes that the terms missing in this last equation correspond to the terms in equation (5.34) with a minus sign. However, such negative terms can occur naturally in the K-theory approach as used in [32] but can never occur in a tensor product decomposition.

We conclude that fusion rules of the skeleton group are to some extent consistent with the OPE algebra discussed by Kapustin and Saulina. The advantage of their approach is first that there is never need to restrict the gauge groups to certain subgroups as we effectively do with the skeleton group. Also, the OPEs of Wilson-'t Hooft operators do indeed give a unified electric-magnetic algebra, whereas in the skeleton group approach one does still need the dual skeleton group. Nonetheless, because of the occurrence of negative terms the OPE algebra cannot be interpreted as a set of physical fusion rules for dyons. In section 7.5 we shall therefore use our skeleton group approach to investigate non-abelian phases with dyonic condensates.

6. S-duality

To check the validity of the skeleton group we shall show that the standard S-duality action on the complex coupling of the gauge theory and the electric-magnetic charges is respected by the skeleton group. We shall first recapitulate some details of S-duality. Second, we discuss its action on the dyonic charge sectors and finally we prove that there is a well-defined S-duality action on the skeleton group representations.

6.1 S-duality for simple Lie groups

In $\mathcal{N}=4$ SYM theory S-duality leaves the BPS mass invariant. The universal mass formula for BPS saturated states in a theory with gauge group G can be written as [33, 15]:

$$M_{(\lambda,g)} = \sqrt{\frac{4\pi}{\operatorname{Im}\tau}}|v\cdot(\lambda+\tau g)|. \tag{6.1}$$

The electric charge λ takes value in the weight lattice $\Lambda(G)\subset\mathfrak{t}^*$ while g is an element in the weight lattice $\Lambda(G^*)\subset\mathfrak{t}$ of the GNO dual group. The complex coupling τ is defined as

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \tag{6.2}$$

The action of S-duality groups on the electric-magnetic charges is discussed by Kapustin in [15], see also [34, 33]. First we choose the short coroots to have length $\sqrt{2}$, i.e.

$$\langle H_\alpha, H_\alpha \rangle = 2. \tag{6.3}$$

Now define a map ℓ acting on the CSA of G and its dual

$$\begin{aligned} \ell : H_\alpha \in \mathfrak{t} \mapsto H_\alpha^\star &= \frac{\langle H_\alpha, H_\alpha \rangle}{2} \alpha \in \mathfrak{t}^* \\ \ell^{-1} : \alpha \in \mathfrak{t}^* \mapsto \alpha^\star &= \frac{2 H_\alpha}{\langle H_\alpha, H_\alpha \rangle} \in \mathfrak{t}. \end{aligned} \tag{6.4}$$

This map is implicitly used in the definition of the BPS mass formula since

$$v \cdot H_\alpha \equiv \frac{\langle H_\alpha, H_\alpha \rangle}{2} v \cdot \alpha \tag{6.5}$$

which indeed leads to the usual degeneracy in the BPS mass spectrum.

Now consider the following actions of the generators:

$$C : \tau \mapsto \tau \qquad (\lambda, g) \mapsto (-\lambda, -g) \tag{6.6}$$

$$T : \tau \mapsto \tau + 1 \qquad (\lambda, g) \mapsto (\lambda - g^\star, g) \tag{6.7}$$

$$S : \tau \mapsto -\frac{1}{\tau} \qquad (\lambda, g) \mapsto (g^\star, -\lambda^\star). \tag{6.8}$$

One can check that $C^2 = 1$, $S^2 = 1$ and $(ST)^3 = C$. It should be clear that T and S generate $SL(2, \mathbb{Z})$ and that C is the non-trivial element of its center. Moreover, one can easily verify that the action of these generators leaves the BPS mass formula (6.1) invariant. Unfortunately, the electric-magnetic charge lattice $\Lambda(G) \times \Lambda(G^*)$ is in general not invariant under the action of $SL(2, \mathbb{Z})$. However, as explained in section 3, it is natural in an $\mathcal{N} = 4$ gauge theory with smooth monopoles to take both G and G^* to be adjoint groups and thereby restrict the electric charges to the root lattice and the magnetic charges to the coroot lattice. One can show that lattice $\Lambda_r \times \Lambda_{cr}$ is invariant under some subgroup of $SL(2, \mathbb{Z})$. To start we note that a long coroot H_α is mapped to a multiple of α since the length-squared of a long coroot is an integral multiple of the length-squared for a short coroot. Consequently, the image of Λ_{cr} under ℓ is contained in the root lattice Λ_r of G. Next we need to check if ℓ^{-1} maps the root lattice of G into the coroot lattice. Note that the long roots are mapped on the short coroots. This means that the length-squared of the image of a short roots has length-squared smaller than 2. Hence the root lattice is mapped into the coroot lattice by ℓ^{-1} only if G is simply-laced.

One finds that the action of the generator S does not leave $\Lambda_r \times \Lambda_{cr}$ invariant in the non-simply laced case, but even then one can still consider the transformation ST^qS which acts as

$$ST^qS : (\lambda, g) \to (-\lambda, -q\lambda^\star - g). \tag{6.9}$$

For q sufficiently large $q\lambda^\star$ is always an element of the coroot lattice, hence there is a subgroup $\Gamma_0(q) \subset SL(2, \mathbb{Z})$ that generated by C, T and ST^qS that leaves $\Lambda_r \times \Lambda_{cr}$ invariant. The largest possible duality group for e.g. $SO(2n+1), Sp(2n)$ and F_4 is $\Gamma_0(2)$ while for G_2 it is $\Gamma_0(3)$.

6.2 S-duality on charge sectors

We have seen above that there is an action of $SL(2,\mathbb{Z})$ or at least an action of a subgroup $\Gamma_0(q)$ if we restrict the electric-magnetic charge lattice to $\Lambda_r \times \Lambda_{cr}$. The restriction of the charge lattice also defines a restriction of the dyonic charges sectors to $(\Lambda_r \times \Lambda_{cr})/\mathcal{W}$. Here we shall show that the duality transformations give a well defined action on these charge sectors.

The generators of the duality group may map (λ, g) to a different equivalence class under the action of the Weyl group and hence to a different charge sector. However, the duality transformations map a Weyl orbit to a Weyl orbit as follows from the fact that the action of the generators of $SL(2,\mathbb{Z})$ commute with the diagonal action of the Weyl group [15]. This is obvious for C since $wC(\lambda,g) = w(-\lambda,-g) = (-w(\lambda),-w(g)) = Cw(\lambda,g)$. For T and $w \in \mathcal{W}$ we have: $wT(\lambda,g) = w(\lambda + g^\star, g) = (w(\lambda) + w(g^\star), w(g)) = (w(\lambda) + w(g)^\star, w(g)) = T(w(\lambda), w(g)) = Tw(\lambda,g)$. Finally for S we have $wS(\lambda,g) = w(-g^\star,\lambda^\star) = (-w(g)^\star, w(\lambda)^\star) = Sw(\lambda,g)$.

6.3 S-duality and skeleton group representations

Since there is a consistent action of the duality group on the dyonic charge sectors one may also try to extend this action to the set of representations of the skeleton group which are labelled by the dyonic charge sectors and by centraliser representations of the lifted Weyl group W. We shall assume that the duality action does not affect the centraliser labels. This is consistent if, one, it maps an irreducible representation to another irreducible representation and if, two, the action respects the fusion rules. Note that we are not considering all representations of the skeleton group but only those that correspond to the root and coroot lattice. Effectively we thus have modded the skeleton group out by a discrete group.

To prove the consistency of the duality group action we shall use the following ingredients. First, the action of C, T and S, and hence also the action of the duality group commutes with the action of the lifted Weyl group. This follows immediately from the fact that the duality group commutes with the Weyl group as we have shown in the previous section. Second, the centraliser subgroup in W is invariant under the action of the duality group on the electric and magnetic charge.

Since the action of $\mathcal{W}$ and thus also W on the electric-magnetic charges is linear it should be clear that charge conjugation does not change the centraliser. The fact that T leaves the centraliser group $W_{(\lambda,g)} \subset W$ invariant is seen a follows: let $W_g \subset W$ be the centraliser of g so that for every $w \in W_g$ $w(g) = g$. The centraliser of (λ, g) consists of elements in $w \in W_g$ satisfying $w(\lambda) = \lambda$. Similarly the elements $w \in W_{(\lambda+g^\star,g)}$ satisfy $w(g) = g$ and thus $w(g^\star) = g^\star$. Finally one should have $w(\lambda + g^\star) = \lambda + g^\star$. But since $w(\lambda + g^\star) = w(\lambda) + w(g^\star)$ one finds that w must leave λ invariant. Hence $W_{(\lambda+g^\star,g)} = W_\lambda \cap W_g = W_{(\lambda,g)}$. Similarly the action of S is seen to leave the leave $W_{(\lambda,g)}$ invariant since $W_{\lambda^\star} = W_\lambda$ and $W_{-g^\star} = W_g$ so that $W_{-g^\star} \cap W_{\lambda^\star} = W_\lambda \cap W_g$.

An irreducible representation of the skeleton group is defined by an orbit in the electric-magnetic charge lattice and an irreducible representation of the centraliser in W of an element in the orbit. Since the $SL(2,\mathbb{Z})$ action commutes with the action of the lifted Weyl group, a W orbit is mapped to another W orbit. We define the centraliser representation to be invariant under the duality transformation. This is consistent because the centraliser subgroup itself is invariant under $SL(2,\mathbb{Z})$. We thus find that an irreducible representation of the skeleton group is mapped to another irreducible representation under the duality transformations.

Finally we prove that S-duality transformations respect the fusion rules of the skeleton group. The claim is that if for irreducible representations Π_a of the skeleton group one has

$$\Pi_a \otimes \Pi_b = n_{ab}^c \Pi_c, \tag{6.10}$$

then for any element s in the duality group one should have

$$\Pi_{s(a)} \otimes \Pi_{s(b)} = n_{ab}^c \Pi_{s(c)}. \tag{6.11}$$

By inspecting equation (5.26) we can prove this equality. First we note that since s commutes with the lifted Weyl group we have for any $(\mu', h') \in [s(\lambda, g)]$ $(\mu', h') = s(\mu, h)$ for a unique $(\mu, h) \in [\lambda, g]$. In that sense the summation over the orbits $[\lambda, g]$ and $[s(\lambda, g)]$ is equivalent. Next we see that since s is an invertible linear map on the dyonic charges $s(\mu_3, g_3) = s(\mu_1, g_1) + s(\mu_2, g_2)$ if and only if $(\mu_3, g_3) = (\mu_1, g_1) + (\mu_2, g_2)$. Similarly we find $hs(\mu, g) = s(\mu, g)$ if and only if $h(\mu, g) = (\mu, g)$. Finally we note that if one defines $x_{(\mu,h)} \in W$ by $x_{(\mu,h)}(\lambda, g) = (\mu, h)$ then $x_{(\mu,h)} s(\lambda, g) = s(x_{(\mu,h)}(\lambda, g)) = s(\mu, h)$ and hence $x_{s(\mu,h)} = x_{(\mu,h)}$. With our convention that the S-duality action does not affect the centraliser charges we now find

$$\langle \chi_c, \chi_{a\otimes b} \rangle = \left\langle \chi_{s(c)}, \chi_{s(a)\otimes s(b)} \right\rangle . \tag{6.12}$$

This proves (6.11).

7. Gauge Fixing and non-abelian phases

A property of the skeleton group is that it does not explicitly incorporate the full non-abelian electric symmetry in magnetically neutral sectors. This is because the skeleton group is actually the effective symmetry in a certain gauge which we call the skeleton gauge. Since the dyonic charge sectors, and hence the skeleton group, were initially defined by rotating the magnetic charges into the CSA, the appearance of such a gauge at this stage is not very surprising. More important is the fact that the skeleton gauge is an example of a non-propagating gauge. Such gauges and in particular the so-called abelian gauge have been introduced by 't Hooft [17]. The skeleton gauge is a minimal non-abelian extension of the abelian gauge that adds (lifted) Weyl group transformations to the residual abelian symmetry of the abelian gauge.

A compelling conclusion of 't Hooft's is that in the gauge fixing procedure smooth topologically non-trivial configurations, such as 't Hooft-Polyakov monopoles [35, 36], lead to singularities which have to be taken into account as dynamical degrees of freedom of the effective theory in the non-propagating gauge. Now, the main advantage of the skeleton gauge is that it can be applied to certain topologically non-trivial field configurations known as Alice strings [22, 23, 24], whereas the abelian gauge fails in these situations. The Alice strings can thus consistently be taken into account in the effective theory if the skeleton gauge is applied. In the case that G equals $SO(3)$ we shall enumerate the singularities that may appear naturally in a non-propagating gauge and discuss which of these singularities obstruct the implementation of the abelian gauge or the skeleton gauge.

The significance of non-propagating gauges is that, if chosen appropriately, they may highlight the relevant degrees of freedom, at large, or better intermediate distance scales. The effective theory corresponding to a certain non-propagating gauge can be very suitable for describing a particular phase of the original theory. The abelian gauge for example is related in this way to the Coulomb phase where the long-range forces are indeed abelian, while the skeleton gauge turns out to correspond to a generalised Alice phase.

Since physical observables are gauge independent one may also use a non-propagating gauge to study other phases the most prominent of which are confining phases. The beauty of ’t Hooft’s approach is that it (at least qualitatively) clarifies confinement in non-abelian gauge theories by exploiting the fact that a strongly coupled abelian theory with monopoles does indeed confine through the condensation of monopoles [37, 38]. Approximate models of this sort have been successfully implemented in certain lattice formulations to investigate the confining phase of $SU(2)$ theories [20]. Generalising this philosophy we set a first step in investigating non-abelian phases which emerge from generalised Alice phases. Note in particular that the skeleton group allows us to study non-abelian phases corresponding to dyonic condensates.

7.1 The abelian gauge and the skeleton gauge

To appreciate the relevance of the skeleton gauge one needs some understanding of the abelian gauge and non-propagating gauges in general. In ’t Hooft’s proposal [17] a non-propagating gauge is introduced by means of some tensor X transforming in the adjoint representation of the gauge group. For G equal to $SO(3)$ one can express X as

$$X = \eta_i \sigma_i, \tag{7.1}$$

where $(\sigma_i)_{i=1,\dots,3}$ are the Pauli matrices. Note that the stabiliser of X equals $U(1) \subset SO(3)$ unless X vanishes. The gauge fixing parameter X can either be a fundamental field of the theory or be defined in terms of a composite field. In a pure Yang-Mills theory one can take for example X to be contained in the tensor product $F_{\mu\nu} \otimes F^{\mu\nu}$. Note that in the case that G equals $SU(2)$ or $SO(3)$ the decomposition of the symmetric tensor product of the adjoint representation into irreducible representations does not contain the adjoint representation itself. For such a pure Yang-Mills theory one thus needs some other field to define X.

One can now fix a gauge by requiring X to be a diagonal matrix, i.e. by requiring X to take values in the CSA. However, to obtain the abelian gauge one also has to fix the order of the eigenvalues. In the case of $SU(n)$ we may for example require X to be of the form $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n$. Since the eigenvalues are gauge invariant the abelian gauge is the strongest non-propagating gauge condition that can be implemented by means of X. If we leave out the additional constraint we obtain a non-propagating gauge condition where the eigenvalues are ordered up to Weyl transformations. This means that for generic configurations with non-vanishing values of X the redundancy of the theory is not restricted to abelian gauge transformations but also contains gauge transformations that correspond to the Weyl group. For e.g. $G = SO(3)$ this residual gauge symmetry is thus the electric subgroup of the skeleton group $O(2) \sim \mathbb{Z}_2 \ltimes U(1)$, i.e. it is a minimal non-abelian extension of the maximal torus $SO(2) \sim U(1)$.

It is not very difficult to see that also for general gauge groups the residual gauge symmetry equals the electric subgroup of the skeleton group if the ordering condition of the abelian gauge is dropped. For this weaker gauge condition the residual gauge symmetry is given by the maximal torus and in addition a discrete group which does not commute with T. The elements of this discrete group are elements in G acting on the eigenvalues of X as the Weyl group, i.e. they are elements in the lift W of $\mathcal{W}$. The total residual gauge group coincides with $(W \ltimes T)/D$, which is by definition the electric subgroup of the skeleton group. Therefore we can identify this extension of the abelian gauge as the skeleton gauge.

Another way to implement the skeleton gauge for $SO(3)$ is not to use a gauge fixing parameter in the adjoint representation but instead a gauge fixing parameter Y in the 5-dimensional irreducible representation. One can identify such a parameter Y with the symmetric traceless tensor

$$Y_{ij} = \eta_i \eta_j - \frac{1}{3}\delta_{ij}\eta_k\eta_k. \tag{7.2}$$

If η is nonzero the $U(1)$ group of rotations around the vector η leave by definition η invariant and hence Y invariant. The latter is also invariant under the Weyl reflection, which is realised in $SO(3)$ as a 180 degree rotation about an axis orthogonal to η and therefore maps $\eta \mapsto -\eta$. If Y does not vanish it breaks $SO(3)$ to $O(2) \sim \mathbb{Z}_2 \ltimes U(1)$. The skeleton gauge is thus manifestly implemented by requiring η to be diagonal. This non-propagating gauge is the strongest gauge condition that can be implemented by means of Y as follows from the fact that it is invariant under Weyl reflections and the fact that eigenvalues of η are gauge invariant. Also note that the gauge fixing parameter Y can be used in a pure $SU(2)$ or $SO(3)$ Yang-Mills theory because, as opposed to the adjoint representation, the 5-dimensional representation appears in the decomposition of the symmetric part of $3 \otimes 3$.

Also for general gauge groups the skeleton group can be implemented with a gauge fixing parameter that, in contrast to X, is actually invariant under the action of the Weyl group for all configurations. A suitable tensor which generalises the symmetric traceless tensor Y_{ij} used in the $SO(3)$ case is described as follows: if λ is the highest weight of the adjoint representation of G then Y should transform in the irreducible representation with highest weight 2λ. The connection with the $SO(3)$ case is seen from the fact that the 2λ representation is part of the symmetric tensor product $\mathrm{Sym}(\lambda \otimes \lambda)$. Moreover, to prove that for generic values this tensor does indeed break G to S_{el} one can use the fact that each $SU(2)$ or $SO(3)$ subgroup corresponding to a simple root is broken down to respectively $(\mathbb{Z}_4 \ltimes U(1))/\mathbb{Z}_2$ or $\mathbb{Z}_2 \ltimes U(1)$.

7.2 Gauge singularities and gauge artifacts

An important aspect of the abelian gauge is that it gives rise to singularities. To see this one can start out from a configuration of X, not necessarily corresponding to a solution of the classical equations of motion, that for the sake of simplicity is smooth over $\mathbb{R}^3$, i.e. X is a section of a trivial adjoint bundle associated to a trivial principal G bundle over $\mathbb{R}^3$. One may now wonder if for such a configuration there always is a smooth or at least continuous gauge transformation that rotates X into the CSA with a fixed order of eigenvalues. If this can be done one ends up with trivial T bundle over $\mathbb{R}^3$ and in particular with a trivial T bundle over any sphere $S^2 \subset \mathbb{R}^3$. We know already that there are configurations corresponding to a trivial G bundle over $\mathbb{R}^3$ for which such a gauge transformation does not exist. These are related to non-trivial T bundles over a 2-sphere.

An example for $G = SU(2)$ (or $G = SO(3)$) directly related to the 't Hooft-Polyakov monopole [35, 36] is if X equals the "hedgehog" configuration $r_i \sigma_i h(r)$ with $h(r)$ approaching some constant value for small values of r. Note that the stabiliser of X at each point in $\mathbb{R}^3$ is a subgroup $U(1) \subset SU(2)$ generated by $\hat{r}_i \sigma_i$, except at the origin where X vanishes and the residual gauge group is restored to $SU(2)$. There is a gauge transformation that maps the hedgehog configuration to $\sigma_3 r h(r)$ which is discontinuous along the negative z-axis (including the origin). This Dirac string is just a gauge artifact as can be seen by adopting the Wu-Yang description [39]; there exists another gauge transformation mapping $\hat{r}_i \sigma_i$ to σ_3 which is discontinuous on the positive z-axis. These two $SU(2)$ gauge transformations are related by a non-trivial $U(1)$ gauge transformation which is well-defined on $\mathbb{R}^3$ except for the z-axis. Consequently the hedgehog configuration defines a non-trivial $U(1)$ bundle on each S^2 centred around the origin. The Dirac strings are now accounted for by using a separate gauge transformation on each of the two hemispheres of S^2. Nonetheless, the patched gauge transformation on $\mathbb{R}^3$ is still singular at the origin where the full gauge group is restored.

In general there exist smooth configurations of X which define a non-trivial winding number in $\pi_2(SU(2)/U(1)) \sim \pi_1(U(1))$ and can therefore not be rotated into the CSA without introducing point-like singularities. What one learns from 't Hooft [17] is that all these singularities have to be added as extra degrees of freedom to the effective theory in the abelian gauge.

It is reasonable to ask if there are not any other types of singularities which one has to add to the effective theory. We have already seen that there are string-like objects such as the Dirac string which are merely gauge artifacts. An alternative way to see this is that the state of any particle in the theory remains unchanged if the particle is moved around a Dirac string. Alice fluxes [23, 24] on the other hand are string-like objects which are truly physical; the state is transformed by a non-trivial Weyl transformation if the particle is moved around the string. The appearance of such a string-like object was in some sense already foreseen by 't Hooft, see section 3 of [17]. We discuss these fluxes in more detail in section 7.3.

Say we start out from a particular configuration corresponding to a smooth Alice flux

[40, 41] so that in cylindrical coordinates we have

$$\eta = \hat{\eta}_i(r,\theta)\sigma_i\alpha(r) \tag{7.3}$$

with

$$\hat{\eta}_i(r,\theta) = R_{ij}(\theta/2)\hat{\eta}_j(r,0), \tag{7.4}$$

where $\hat{\eta}(r,0) = \sigma_3$ for large values of r, $\hat{\eta}(r,0) = \sigma_1$ for $r \to 0$ and $R_{ij}(\varphi)$ corresponds to a rotation around the x-axis over an angle φ. The $U(1)$ factor of the residual gauge group that leaves η invariant at some point on an infinitely large cylinder is generated by $\hat{\eta}(\theta) := \lim_{r\to\infty} \hat{\eta}(r,\theta)$. If one goes around the flux this generator is transformed by a $\mathbb{Z}_2$ action since $\hat{\eta}(2\pi) = -\hat{\eta}(0)$.

Let us now assume that, in accordance with our definition in section 7.1, X indeed equals η as given directly above. There certainly exists a gauge transformation that rotates X into the CSA. This gauge transformation, which is essentially given by the the rotation matrix $R(\theta/2)$, is discontinuous at $\theta = 0$ and thus gives rise to a plane-like singularity bounded by the z-axis. Again this singularity is just a gauge artifact that can be circumvented by using a two-patched gauge transformation. One now obtains a non-trivial bundle on each cylinder centred around the z-axis which is essentially a Möbius strip. But even in this description there remains a singularity at the z-axis. Since Alice fluxes mediate topological interactions, see e.g. [42], their corresponding string singularities should be taken into account in the effective theory.

It is nice to note that at the string singularity the full $SU(2)$ group is not restored as happens at the hedgehog singularity. Instead, the stabiliser of X equals a $U(1)$ group as follows from the fact that in the smooth Alice configuration defined above, X becomes proportional σ_1 at the z-axis [41]. What is crucial, however, is that X is not single-valued if we take it around the flux. This means that even though for an Alice flux configuration X can be rotated into the CSA, its eigenvalues cannot be ordered since they are permuted as one goes around the flux; both orderings appear simultaneously. Consequently the abelian gauge cannot be implemented if X corresponds to an Alice flux configuration, the strongest gauge condition that can be used is the skeleton gauge in which the ordering condition is left out and the gauge is fixed only by taking X to be diagonal.

One may object to our conclusion that since X is not single-valued for an Alice flux configuration, such a configuration is not allowed anyway, even not before gauge fixing. However, one should not confuse a possible configuration of X in the background of an Alice flux with a smooth Alice flux tube itself. The latter is certainly a perfectly smooth as well as single-valued configuration of the magnetic field just as a smooth magnetic monopole. What we actually observe here is that an adjoint field, such as for example X, in the background of an Alice flux is indeed not necessarily single-valued on a circle enclosing the Alice string. This fact does obstruct the implementation of the abelian gauge. If one still insists on using the abelian gauge this means one has to disregard all field configurations with an Alice flux background thereby ignoring a relevant sector of the theory in which its non-abelian nature is manifest. These observations will hopefully become more clear in section 7.3 where we discuss the Alice flux tube in some detail.

In contrast to the abelian gauge, the skeleton gauge can still be applied in the background of an Alice flux since in the skeleton gauge one does not require the eigenvalues of X to be ordered in a given way. A particular nice way the implement the skeleton gauge for $SO(3)$ is to use the gauge fixing parameter Y corresponding to the 5-dimensional irreducible representation as discussed in section 7.1. Since Y is invariant under the Weyl group action it follows that this gauge fixing parameter is single-valued in an Alice flux background. In particular, the Alice flux configuration of Y defined by equations (7.2), (7.3) and (7.4), as opposed to the related Alice flux configuration of X, is manifestly smooth over $\mathbb{R}^3$.

General smooth configurations of Y which correspond to a non-trivial equivalence classes in $\pi_1(SO(3)/(\mathbb{Z}_2 \ltimes U(1)) \sim \pi_0(\mathbb{Z}_2)$ give rise to string-like singularities which have to be added to the effective theory in the skeleton gauge. Also point-like singularities appearing in the skeleton gauge have to be taken into account. The smooth configurations from which these singularities arise define non-trivial elements in $\pi_2(SO(3)/(\mathbb{Z}_2 \ltimes U(1)) = \pi_1(\mathbb{Z}_2) \times \pi_1(U(1))$. Since $\pi_1(\mathbb{Z}_2)$ is trivial such point-like singularties are directly related to the monopole singularities in the abelian gauge with the difference that the sign of the monopole charge is only defined up to a sign in an Alice flux background. This last conclusion follows from the heuristic argument that the field strength tensor $F_{\mu\nu}$, and hence also the magnetic field B_i corresponding to a monopole in an Alice flux background is not single-valued on a circle enclosing the Alice string. More precise arguments for this are based on topology [42].

Besides point-like and string-like singularities there might also be plane-like singularities in the skeleton gauge or in the abelian gauge. We have already seen such singularities in the skeleton gauge which correspond to a discontinuity of the $U(1)$ generator in the flux background. Such singularities can only be truly physical if they arise from a smooth configuration defining a non-trivial element in $\pi_0(G/H)$ for some subgroup in $H \subset G = SO(3)$. This homotopy group vanishes in the case we consider here as follows from the fact that $SO(3)$ is connected. That tells us that all plane-like singularities must be gauge artifacts.

Just as the abelian gauge cannot always implemented for Alice flux configurations there may be configurations which obstruct the implementation of the skeleton gauge. Such configurations do indeed exist and correspond for example to topologically stable flux solutions in theories where the gauge group is broken to a discrete (non-abelian) subgroup of $SO(3)$ which is not a subgroup of $\mathbb{Z}_2 \ltimes U(1)$. If one goes around such a flux, $\eta = \eta_i \sigma_i$ is transformed to $\mathrm{Ad}(h)(\eta)$ with $h \notin \mathbb{Z}_2 \ltimes U(1)$ and $\eta \sim \sigma_3$ is mapped out of the CSA. Note, however, that a suitable gauge fixing parameter that is single-valued in the presence of such a flux, transforms in some higher dimensional representation of the gauge group and can in general thus not be constructed as a composite field in a pure Yang-Mills theory. Moreover, even if a suitable parameter can be constructed or if there is a fundamental field transforming in the appropriate representation, monopoles will be confined in such a flux sector because the flux configuration corresponds to an electric condensate that manifestly breaks $U(1) \subset SO(3)$. We thus have to conclude that even though the skeleton gauge does not probe all sectors of the theory and thereby does not show the complete non-abelian symmetry, it should be sufficient to describe all dyonic charge sectors.

7.3 Generalised Alice phases

The effective theory in the abelian gauge is particularly suitable for describing the Coulomb phase. Similarly we claim that the skeleton gauge corresponds to a generalised Alice phase. In this section we review some well known properties of the prototype of an Alice phase involving $SO(3)$ and finally generalise to arbitrary gauge groups.

The Alice phase for $SO(3)$ can be described as a Higgs phase with a condensate in the 5-dimensional irreducible representation of $SO(3)$ [22, 23, 24]. The expectation value of the Higgs field Φ can be identified with the traceless symmetric tensor

$$\langle \Phi_{ij} \rangle = \eta_i \eta_j - \frac{1}{3} \delta_{ij} \eta_k \eta_k. \tag{7.5}$$

This is the same expression as for the gauge fixing parameter Y in equation (7.2). It follows from the discussion there that the residual symmetry group for the case $\eta \neq 0$ is $\mathbb{Z}_2 \ltimes U(1)$, i.e. the electric subgroup of the skeleton group for $SO(3)$.

Just as the Coulomb phase resulting from an adjoint Higgs condensate, the Alice phase allows for monopoles because $\pi_2(G/H) = \mathbb{Z}$, but it also allows a non-trivial $\mathbb{Z}_2$ flux because $\pi_1(G/H) = \mathbb{Z}_2$. A smooth $\mathbb{Z}_2$ flux solution has been constructed in [40], see also [41]. The ansatz for this solution is due to Schwarz [23]. In cylindrical coordinates the asymptotic Higgs field $\Phi(r, \theta, z) = \phi(\hat{r}) + \mathcal{O}(r^{-1})$ is determined in terms of $\eta = \hat{\eta}(\theta)\alpha(r)$ via equation (7.5) by

$$\hat{\eta}_i(\theta) = R_{ij}(\theta/2)\hat{\eta}_j(0), \tag{7.6}$$

where $\eta(0) \sim \sigma_3$ and $R_{ij}(\varphi)$ corresponds, as in the previous section, to a rotation around the x-axis over an angle φ. It is important to note that this flux solution itself is perfectly smooth at $\theta = 0$. As a matter of fact the Higgs field and actually also the gauge field are by construction perfectly smooth on $\mathbb{R}^3$.

In general one can associate to a flux solution a specific element h in the residual gauge group defined by a Wilson loop around the flux:

$$h = Pe^{\oint A_\mu dx^\mu}. \tag{7.7}$$

The element h can in principle be any element in H, but the flux is topologically stable if it lies in a connected component of H which does not contain the unit. In this particular case we thus find that up to a $U(1)$ element this so-called Alice flux h equals the generator of $\mathbb{Z}_2$. This implies that if an electrically charged particle is moved around the Alice flux, its charge is conjugated by the $\mathbb{Z}_2$ action. The important conclusion is that it is not possible to give a single-valued definition for the electric $U(1)$ charges in the presence of an Alice flux. In other words the $U(1)$ generator is not single-valued and changes sign if one takes it around the $\mathbb{Z}_2$ flux as we already discussed in section 7.2. In that sense the long range interaction in the Alice phase are described by an abelian theory with a local charge conjugation symmetry which is known as Alice electrodynamics. This theory is locally not different from ordinary electrodynamics describing the long range interactions of the Coulomb phase. Nonetheless, on a global level these theories are profoundly different because the Alice fluxes mediate topological interactions, see e.g. [42].

The Alice phase of $SO(3)$ as well as the skeleton gauge are easily generalised to other gauge groups. A generalised Alice phase is per definition a gauge theory with gauge group G broken to its electric skeleton group S_{el} or some non-abelian subgroup of S_{el}. Even though the residual gauge group is non-abelian the effective theory in such a phase should still be "manageable" since the non-abelian factor is only manifest via topological interactions mediated by Alice fluxes. The latter correspond to disconnected components of the skeleton group. Note that two elements in the lift of the Weyl group $W \subset S_{el}$ are connected if they differ by an element in T. Hence the Alice fluxes can be labelled by elements in the Weyl group $\mathcal{W} = W/D$ of G, where $D = W \cap T$.

The appropriate Higgs field which generalises the symmetric traceless Higgs used in the $SO(3)$ case corresponds to the same representation as the gauge fixing parameter Y introduced in section 7.1. If one wants to prove that such a Higgs may indeed break G to S_{el} one should use the same arguments as used in this preceding section.

7.4 Unified electric-magnetic descriptions

In order to understand non-perturbative phenomena of a non-abelian gauge theory by means of the effective theory in some non-propagating gauge it is essential to give a complete characterisation of this effective theory. This is illustrated for example by the observation that in the abelian gauge the condensation of point-like degrees of freedom corresponding to magnetic monopoles leads to electric confinement. Similarly the string-like singularities corresponding to Alice fluxes which appear in the skeleton gauge have to be included in the description of the effective theory in that gauge. An a posteriori justification for this conclusion can be found in section 7.5 where we briefly discuss the impact of the condensation of the Alice strings.

One additional element needed in the characterisation of the effective theory is its symmetry, including the magnetic component. The motivation for this is that magnetic condensates actually correspond to representations of a magnetic group just as electric condensates are related to a representation of an electric gauge group. As long as one wants to deal with purely electric or purely magnetic condensates only, it is sufficient to use the appropriate subgroups of G and G^*. However, to study dyonic condensates in the effective theory one is forced to use the dyonic fusion rules defined by the unified electric-magnetic symmetry of the effective theory.

The only conceivable candidate for the total symmetry of the effective theory in the abelian gauge is the product group $T \times T^*$, while for the skeleton gauge the obvious candidate is the skeleton group. One could even try to go further and argue that each of these unified symmetries is actually the gauge symmetry of the effective theory in respectively the abelian gauge and the skeleton gauge.

To illustrate this we consider an abelian gauge theory with monopoles. This theory has a manifest electric gauge group T, which we take to be $U(1)$ for example, and a topological conservation law for the magnetic charge. To account for the magnetic part of the spectrum we may assume the theory to have a hidden global $T^* = U(1)$ whose set of charges is in one-to-one relation with the set of topological charges $\pi_1(T)$. The magnetic condensates in this theory are thus labelled by irreducible representations of T^*.

One may also introduce a second gauge potential making T^* local, but then one has to impose a constraint to ensure that the second gauge field does not carry physical degrees of freedom, as the theory has only a single photon. This can be done and was formulated in e.g. [43] in an attempt to make a consistent quantum field theory involving both electrically charged particles and magnetic monopoles. So the conclusion is that an abelian gauge theory with monopoles can be viewed as a constrained $T \times T^*$ gauge theory. This holds in particular for the effective theory describing an $SO(3)$ Yang-Mills theory in the abelian gauge.

Next we consider Alice electrodynamics, i.e. a gauge theory with a manifest electric gauge group $\mathbb{Z}_2 \ltimes U(1)$ with additional magnetic monopoles and Alice fluxes. Again there is a topological conservation law for the magnetic charges as well as one for the Alice fluxes. The set of monopole charges is identical to the set of monopoles in the abelian theory where the residual symmetry equals $U(1)$. The difference with the latter case lies in the fact that because of the presence of Alice fluxes, monopoles with charges related by Weyl transformations should be identified. A heuristic argument for this is that the magnetic charge is defined in terms of the $U(1)$ generator which is not single-valued in the background of an Alice flux as discussed in section 7.3 and thus the sign of the magnetic charge flips if we take it around the $\mathbb{Z}_2$ flux. More precise arguments for this are based on topology [42]. It should now also be clear that if we take a dyon around an Alice flux the electric as well as the magnetic charge changes sign, hence the charge conjugation symmetry of Alice electrodynamics acts diagonally on dyonic charges. These properties are accurately reflected by the skeleton group. We thus find that the skeleton group $\mathbb{Z}_2 \ltimes (T \times T^*)$ is the obvious candidate for the unified electric-magnetic symmetry of Alice electrodynamics.

We now claim that the effective theory for the Yang-Mills theory with gauge group G in the skeleton gauge is a gauge theory generalising Alice electrodynamics. This effective theory is up to (lifted) Weyl transformations an abelian theory with additional monopoles. The non-abelian character of the theory is locally invisible and can only be observed in topological interactions mediated by Alice strings. Generalising the conclusions above we find that the effective theory in the skeleton gauge, and hence the effective theory in a generalised Alice phase, should have a gauge symmetry given by the skeleton group S of G. Note that this also implies all topological features of such effective theories should be computed in terms of this unified gauge group and not in terms of the electric subgroup only.

7.5 Phase transitions: condensates and confinement

In this subsection we want to determine phases of an $SO(3)$ gauge theory related to an Alice phase. Starting from this phase we systematically analyse a number of conceivable condensates and describe the resulting phases. Most common phases are retrieved in this way but already in this simplest example some new results are also obtained. In any case, one encounters an interesting interplay between symmetry group breaking and topological features, leading to an understanding why certain parts of the spectrum are "swallowed" by the vacuum.

Neutral vector boson

Starting from the generalised Alice phase, let us consider the case where the neutral vector particle condenses, i.e. a condensate in the sector $(-,[0,0])$ in the notation introduced in section 5.3. Such a condensate breaks S to $T\times T^*$. This phase is indeed different from the original Alice phase because the Alice string is confined. This means that possible closed loops of Alice string, which had an energy proportional to length in the skeleton phase, become pancakes with an energy proportional to the minimal area spanned by the loop. The topological argument is simple: in the new phase topological domain walls form, these are labelled by $\pi_0(G/H)=\pi_0(S/(T\times T^*))=\pi_0(\mathbb{Z}_2)=\mathbb{Z}_2$, and the Alice loops will become the boundaries of these walls. Since the Alice fluxes disappear from the bulk, opposite electric and magnetic charges are no longer equivalent and therefore the skeleton representations split into $T\times T^*$ representations. This is the Coulomb phase, and as such indeed nothing but the abelian gauge description of the $SU(2)$ theory given by 't Hooft as discussed above.

Charged vector boson

Let us now assume that in addition a charged vector boson in the representation $(2,0)$ of $T\times T^*$ condenses. Note that this condensate breaks $T\subset S$ completely and hence we end up in a Higgs phase. The residual symmetry group is given by T^* and topology is changed since now one has a spectrum of magnetic fluxes corresponding to

$$\pi_1((T\times T^*)/T^*)=\pi_1(T)=\mathbb{Z}. \tag{7.8}$$

This implies that the theory is in a phase where magnetic fluxes are forced into magnetic flux tubes. These flux tubes match the allowed magnetic charges in the theory and one should expect all monopoles to become confined. More precisely: the minimal confined flux equals one fundamental flux quantum corresponding to the fundamental weight of the magnetic dual group $SU(2)$. The derivation of this fact is closely related to the standard derivation of the Dirac quantisation condition for the allowed magnetic charges in terms of the Dirac string. If we move a particle with electric charge $\lambda\in\mathbb{Z}$ around a flux tube with flux $e^{ig\pi}\in U(1)$ the state of the system picks up a phase factor $e^{i\lambda g\pi}$. Hence if we move the charged vector boson around a fundamental flux tube with $g=1$, the state of the system is single-valued and one can consistently think of the vector boson as absorbed by the vacuum. This observation about single-valuedness holds for any particle whose electric-magnetic charge is an integer multiple of $(2,0)$ and hence any such particle may be thought of as being absorbed by the vacuum if a charged vector boson condenses. As a matter of fact the complete purely electric sector may be identified with the vacuum sector if the vector boson condenses. In other words the condensation of the $(2,0)$ representation gives rise to an equivalence relation $(\lambda,0)\sim(0,0)$.

To understand what happens to dyonic charges we first note that after condensation of the charged vector boson the charge sector (λ,g) should be identified with the charge sector $(0,g)$. This follows from the fact that such a dyon originates from the tensor product representation $(\lambda,0)\otimes(0,g)$. Since magnetic monopoles become confined one can now constistently state that any dyon with non-zero magnetic charge gets confined.

Note that a monopole with unit charge will have a single flux tube attached which reaches to infinity, while a monopole with charge 2, corresponding to the root of $SU(2)$, can have two fluxtubes. These observations are the three-dimensional analogue of what is encountered in two-dimensional situations studied in for example [44].

Boundary theory

It is interesting to see what would happen on a boundary. The unit flux in the bulk defines the unit flux on the boundary. In the boundary theory the monopole is an instanton in the sense that tunnelling of a unit flux is allowed via a monopole-antimonopole pair in the bulk. Consequently on the boundary there is no non-trivial magnetic flux sector. Since in the bulk theory all electric charges are condensed, all charge sectors in the boundary theory are identified with the vacuum. The boundary theory is thus trivial.

We shall see that the boundary theory becomes non-trivial if we consider a bulk condensate in the $(4, 0)$ representation which breaks $T \times T^*$ to $\mathbb{Z}_2 \times T^*$. The minimal confined flux is now half the unit flux; if we move a $(4, 0)$ state around this minimal flux the state of the system does not pick up a non-trivial phase factor and the $(4, 0)$ representation can consistently be identified with the vacuum representation $(0, 0)$. Note that this actually holds for any $(4n, 0)$ representation as consistent with the fact that gauge group is broken down to $\mathbb{Z}_2 \times T^*$. We again see that all monopoles are confined but the unit monopole has two flux tubes attached instead of only one, which implies that the minimal confined flux tube cannot break up into a series of monopole-anti monopole pairs. This gives rise to a non-trivial flux sector with half a unit flux. Together with the tunnelling of unit fluxes via monopole-antimonopoles pairs in the bulk this gives rise to a magnetic $\mathbb{Z}_2$ factor of the boundary theory.

To understand the electric content of the boundary theory we first make some observations about the bulk theory. The condensation of electrically charged particles means that at long range their charges are screened. Since all such particles interact via the same Coulomb field we should actually say that this field and thereby all Coulomb interactions are screened at large distances. Nonetheless, as proven in [45], one should expect all topological interactions, mediated by the magnetic fluxes, to survive at long range. If we move a particle with electric charge $2n$ around the minimally confined flux tube, the state of the system picks up the phase factor $e^{\frac{i2\pi n}{2}}$ which is non-trivial for $n = 1$ modulo 2. Particles that can pick up a non-trivial phase factor may interact topologically at long range. Hence, these particles can be distinguished from the vacuum and give rise to the non-trivial charge sector of the boundary theory. The complete picture is that, with a condensate in the $(4, 0)$ representation, on the boundary we have a $\mathbb{Z}_2$ discrete gauge theory [26]. This is consistent with the arguments which one may separately apply to the boundary theory [44].

We should make the following remark: there are phases that are difficult to describe in this language, or better in this gauge. For example the non-abelian phases, corresponding to discrete gauge theories. Certain $\mathbb{Z}_n$ and D_n models are accessible because their group can be embedded in S_{el}. The other electric discrete non-abelian phases can be and have been studied starting from the electric theory [26].

Magnetic condensates

One could also break the T^* by a monopole condensate in say the $(0,2)$ representation of $T \times T^* \subset S$. Now electric flux tubes develop which are quantised and match the possible electric charges in the theory. In this phase one should thus expect that electric charges are confined, i.e. one obtains a confining phase.

What we said about the non-abelian discrete phases applies here, too. The existence of a phase with a non-abelian purely magnetic symmetry has not been proven, though by "undoing" the abelian gauge on the magnetic side one could imagine to obtain such a phase. Our findings in previous sections about the skeleton group and its dual are certainly consistent with this hypothesis.

Dyonic condensates

We also want to see what happens with dyonic condensates. Let us define an exterior product notation between two charge sectors: $[n,m] \wedge [n',m'] = nm' - mn'$. Note that this combination is invariant under the $SL(2,\mathbb{Z})$ action on the pairs of integers (n,m) and (n',m'). If we identify the weight lattice of $SO(3)$ with the even integers we can write the generalised Dirac condition for dyons simply as the condition that the wedge product equals $2k$ for some $k \in \mathbb{Z}$. The condition for confinement given a condensate of $[n,m]$ is now that a sector $[n',m']$ will be confined if $|[n,m] \wedge [n',m']| \geq 2$. Conversely, the only sectors that are not confined are those for which the exterior product with the condensate vanishes, which implies the condition $nm' - mn' = 0$ or $n/m = n'/m'$, in other words the electric-magnetic vectors have to be parallel. These conditions follow from the analogous conditions for a purely condensate via an $SL(2,\mathbb{Z})$ transformation that maps to $[n,m]$ to $[l,0]$. The generalisation to higher rank gauge groups should be similar but now we have the inner products between vectors on Λ and Λ^* in the exterior product: $[\lambda, g] \wedge [\lambda', g'] = \lambda \cdot g' - g \cdot \lambda'$. The condition for confinement that the norm of the exterior product must be larger than or equal to 2 remains roughly the same. Note, however, that the condition for non-confinement allows for many more solutions in this situation.

String condensate

The attentive reader will no doubt have noticed that we have "overlooked" one possible phase that should be part of our analysis. The question is what happens if, in our Alice phase, the Alice strings themselves condense? This possibility has been considered before [46], but the physical implications of such a condensate were not.

The crucial property in the unbroken phase is that electric and magnetic charges can de-localise into so-called Cheshire charges, which are rings of Alice flux carrying a non-localised charge, meaning that any closed surface containing the ring may contain an electric or magnetic charge, but it cannot be localised any further. This makes clear what the situation is like when the Alice strings condense: both the notions of electric and magnetic charge lose their physical meaning. Another way to describe this would be to say that both types of charge can spread and neutralise any source; this means that both electric and magnetic charges will be completely screened. The particles survive as neutral particles. To our knowledge such a phase has never been discussed before and deserves further investigation.

We finally note that for higher rank groups it is very interesting to consider phases corresponding to a condensate that partially breaks the (lifted) Weyl group symmetry. In particular the skeleton group allows one to investigate phases that emerge from subsequent dyonic condensates. Such phases will most likely show some novel features of the underlying non-abelian gauge theory.

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A. Skeleton group for SU(n)

Below we work out the construction of the skeleton group and its irreducible resentations in some detail for $G = SU(n)$. The skeleton groups for the other classical Lie groups are discussed in [47].

We shall start by identifying the lift W of the Weyl group. For the maximal torus T of $SU(n)$, we take the subgroup of diagonal matrices. The length of the roots is set to $\sqrt{2}$. The raising and lowering operators for the simple roots are the matrices given by $(E_{\alpha_i})_{lm} = \delta_{li}\delta_{m,i+1}$ and $(E_{-\alpha_i})_{lm} = \delta_{l,i+1}\delta_{m,i}$. From this one finds that x_{α_i} as defined in equation (4.14) is given by:

$$(x_{\alpha_i})_{lm} = \delta_{lm}(1 - \delta_{li} - \delta_{l,i+1}) + i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{mi}). \tag{A.1}$$

From now on we abbreviate x_{α_i} to x_i. One easily shows that

$$x_i^4 = 1, \qquad [x_i, x_j] = 0 \text{ for } |i-j| > 1, \qquad x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}. \tag{A.2}$$

As it stands, this is not the complete set of relations for W. However, one may show that W is fully determined if we add the relations

$$(x_i x_{i+1})^3 = 1. \tag{A.3}$$

This also makes contact with the presentation of the normaliser of T obtained by Tits [48, 49].

We shall now determine the group D. Note that the elements $x_i^2 \in W$ are diagonal and of order 2. In fact, we have $(x_i^2)_{lm} = \delta_{lm}(1 - 2\delta_{li} - 2\delta_{l,i+1})$. One thus sees that the group K generated by the x_i^2 is just the group of diagonal matrices with determinant 1 and diagonal entries equal to ± 1. Since its elements are diagonal we have $K \subset T$ and hence $K \subset D = W \cap T$. As a matter of fact $K = D$. To prove this one can check that conjugation with the x_i leaves K invariant. Hence K is a normal subgroup of W and thus the kernel of some homomorphism ρ on W. The image of ρ is the Weyl group $\mathcal{S}_n$ of $SU(n)$. To see this note that W/K satisfies the relations of the permutation group (these are the same as the relations for the x_i above, but with $x_i^2 = 1$). An explicit realisation of $\rho : W \to \mathcal{S}_n$ is given by $\rho(w) : t \in T \mapsto wtw^{-1}$. Obviously $D \subset \mathrm{Ker}(\rho) = K$, consequently $D = K$.

Let us work out the $SU(2)$ case as a small example. $SU(2)$ has only one simple root and thus W has only one generator x which satisfies $x^4 = 1$. This gives $W = \mathbb{Z}_4$. D is generated by x^2 which squares to the indentity and hence $D = \mathbb{Z}_2$. For higher rank W is slightly more complicated but D is simply given by the abelian group $\mathbb{Z}_2^{n-1}$.

In order to determine the representations of S for $SU(n)$ we need to solve (5.10) and hence we need to describe how D is represented on a state $|\,\lambda\,\rangle$ in an arbitrary representation of $SU(n)$. This turns out to be surprisingly easy. The generating element x_i^2 of D acts as the non-trivial central element of the $SU(2)$ subgroup in $SU(n)$ that corresponds to α_i. Now let $(\lambda_1, \ldots, \lambda_{n-1})$ be the Dynkin labels of the weight λ. Note that λ_i is also the weight of λ with respect to the $SU(2)$ subgroup corresponding to α_i. Recall that the central element of $SU(2)$ is always trivially represented on states with an even weight while it acts as -1 on states with an odd weight. Hence x_i^2 leaves $|\,\lambda\,\rangle$ invariant if λ_i is even and sends $|\,\lambda\,\rangle$ to $\lambda(x_i^2)|\,\lambda\,\rangle = -|\,\lambda\,\rangle$ if λ_i is odd.

For any given orbit $[\lambda, g]$ we can solve (5.10) by determining $N_{\lambda,g} \subset W$ and choosing a representation of $N_{\lambda,g}$ which assures that the elements (x_i^2, x_i^2) act trivially on the vectors $|\,\lambda, v^\gamma\,\rangle$.

If the centraliser of $[\lambda, g]$ in $\mathcal{W}$ is trivial its centraliser $N_{(\lambda,g)}$ in W equals $D = \mathbb{Z}_2^{n-1}$. An irreducible representations of γ of D is 1-dimensional and satisfies $\gamma(x_i^2) = \pm 1$. The centraliser representations that satisfy the constraint (5.10) are defined by $\gamma(x_i^2) = \lambda(x_i^2)$. If $(\lambda, g) = (0, 0)$ the centraliser is W. In this case an allowed centraliser representation γ satisfies $\gamma(d)|\,v\,\rangle = |\,v\,\rangle$, i.e. γ is a representation of $W/D = \mathcal{W}$. The irreducible representations $\Pi_\gamma^{[0,0]}$ of S thus correspond to irreducible representations of the permutation group $\mathcal{S}_n$. If $N_{(\lambda,g)}$ is neither D nor W the situation is more complicated and we will not discuss this any further.

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